

Erratum to: Idealness of k -wise intersecting families*

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Let \mathcal{C} be a clutter over ground set V . \mathcal{C} is *ideal* if the polyhedron $Q(\mathcal{C}) := \{x \in \mathbb{R}_+^V : \sum_{v \in C} x_v \geq 1 \forall C \in \mathcal{C}\}$ is integral. Given an integer $k \geq 2$, \mathcal{C} is *k -wise intersecting* if every subset of at most k members have a common element, yet no element belongs to all members. In the paper, we studied the following conjecture.

Conjecture 3. *There exists an integer $k \geq 4$ such that every k -wise intersecting clutter is non-ideal.*

Let $n \geq 1$ be an integer, and let $S \subseteq \{0, 1\}^n$. Let $\text{conv}(S)$ denote the convex hull of S . An inequality of the form $\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1$, for some disjoint $I, J \subseteq [n]$, is called a *generalized set covering inequality*. The set S is *cube-ideal* if every facet of $\text{conv}(S)$ is defined by $x_i \geq 0$, $x_i \leq 1$, or a generalized set covering inequality.

Conjecture 16. *There exists an integer $k \geq 4$ such that for every cube-ideal set S , either all the points in S agree on a coordinate, or there is a subset of S of at most k points that do not agree on a coordinate.*

At the beginning of §4 of the paper, we showed that Conjecture 3 for k implies Conjecture 16 for the same k . Then, in §4.2, we proved that the two conjectures are equivalent. While the statement is true, our proof is flawed; we address the source of the flaw shortly. For now, let us provide a shorter, correct proof of this fact.

Theorem 23. *Conjecture 16 for k is equivalent to Conjecture 3 for the same k .*

Revised proof. We already showed (\Leftarrow). It remains to prove (\Rightarrow). Suppose Conjecture 3 is false for some $k \geq 4$. That is, there is an ideal k -wise intersecting clutter \mathcal{C} over ground set $\{1, 2, \dots, n\}$. Let

$$S := \{\chi_C : C \subseteq [n] \text{ contains a member of } \mathcal{C}\}.$$

Observe that S is an up-monotone subset of $\{0, 1\}^n$. By Theorem 4.3 of [2], since \mathcal{C} is an ideal clutter, S is a cube-ideal set. Moreover, since \mathcal{C} is k -wise intersecting, the points in S do not agree on a coordinate but every subset of S of $\leq k$ points do. Therefore, S refutes Conjecture 16 for k , as required. \square

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The culprit. The previous proof of Theorem 23 relied on Theorem 22, but there is an oversight in the proof of that theorem. More specifically, the last two sentences of the proof of Claim 3, starting with “Thus, to finish the proof;” and ending with “We leave this as an easy exercise for the reader.” are incorrect. As a result, we retract the second claim of Theorem 22, as follows.

Theorem 22. *Let \mathcal{C} be an ideal tangled clutter. Then $\text{core}(\mathcal{C})$ is a duplication of a cuboid. ~~Moreover, $\text{core}(\mathcal{C})$ is an ideal tangled clutter.~~*

In particular, we do not prove that every ideal tangled clutter has an ideal core (though this has been proved using more advanced machinery recently [1]). The only result in the paper that relies on Theorem 22 is Theorem 23, which remains correct.

Fixing Theorem 22. The proof of Theorem 22 can be expanded to show something different. Moving forward, we use terminology and results from our paper without recalling them.

Theorem 22a. *Let \mathcal{C} be an ideal tangled clutter over ground set V , let $G = (V, E)$ be the bipartite graph over vertex set V whose edges correspond to the minimum covers of \mathcal{C} , and let $\{U_1, V_1\}, \dots, \{U_r, V_r\}$ be the bipartitions of the connected components of G . Then the following statements hold:*

1. $\text{core}(\mathcal{C})$ is a tangled clutter.
2. For each $i \in [r]$, pick $u_i \in U_i$ and $v_i \in V_i$. Let \mathcal{C}' be the clutter over ground set $\{u_1, v_1, \dots, u_r, v_r\}$ obtained from $\text{core}(\mathcal{C})$ after contracting $V - \{u_1, v_1, \dots, u_r, v_r\}$. Then $\text{conv}(\{\chi_C : C \in \mathcal{C}'\})$ can be described by $z \geq \mathbf{0}$, $z_{u_i} + z_{v_i} = 1, i \in [r]$, and

$$\sum_{B \cap U_i \neq \emptyset} |B \cap U_i| z_{u_i} + \sum_{B \cap V_j \neq \emptyset} |B \cap V_j| z_{v_j} \geq 1$$

for every minimal cover B of \mathcal{C} such that for each $i \in [r]$, $B \cap (U_i \cup V_i) \subseteq U_i$ or V_i .

3. \mathcal{C}' is the cuboid of some subset $S \subseteq \{0, 1\}^r$, where $\text{conv}(S)$ is defined by $\mathbf{0} \leq x \leq \mathbf{1}$, and

$$\sum_{B \cap U_i \neq \emptyset} |B \cap U_i| x_i + \sum_{B \cap V_j \neq \emptyset} |B \cap V_j| (1 - x_j) \geq 1$$

for every minimal cover B of \mathcal{C} such that for each $i \in [r]$, $B \cap (U_i \cup V_i) \subseteq U_i$ or V_i .

Proof. By Proposition 20 (3), for each $i \in [r]$, the elements in U_i are duplicates in $\text{core}(\mathcal{C})$, the elements in V_i are duplicates in $\text{core}(\mathcal{C})$, and $|\{u, v\} \cap C| = 1$ for all $u \in U_i, v \in V_i$ and $C \in \text{core}(\mathcal{C})$. That is, each $C \in \text{core}(\mathcal{C})$ is determined by r binary choices; in each connected component of G , C must contain exactly one of the two parts of the bipartition. This allows a more concise representation of the core. For each $C \in \text{core}(\mathcal{C})$, define $p_C \in \{0, 1\}^r$ such that

$$(p_C)_i = \begin{cases} 0 & \text{if } C \cap (U_i \cup V_i) = V_i \\ 1 & \text{if } C \cap (U_i \cup V_i) = U_i \end{cases}$$

Let $S := \{p_C : C \in \text{core}(\mathcal{C})\} \subseteq \{0, 1\}^r$. Then $\text{core}(\mathcal{C})$ is a duplication of cuboid(S).

Claim 1. $\text{core}(\mathcal{C})$ is a tangled clutter, so (1) holds.

Proof of Claim. As a subset of \mathcal{C} , $\text{core}(\mathcal{C})$ has covering number at most two, and every element of it appears in a cover of cardinality two. Thus, to prove the claim, it suffices to show that $\text{core}(\mathcal{C})$ has covering number at least two. Let y be a fractional packing of \mathcal{C} of value two. Then $\text{support}(y) \subseteq \text{core}(\mathcal{C})$ by Proposition 20 (1), so y is also a fractional packing of $\text{core}(\mathcal{C})$. Subsequently, $\text{core}(\mathcal{C})$ has covering number at least two, as required. \diamond

We know that

$$\{\chi_C : C \in \text{core}(\mathcal{C})\} = \{\chi_C : C \in \mathcal{C}\} \cap \{x : x_u + x_v = 1, \{u, v\} \in E\}. \quad (\star)$$

Claim 2. $\text{conv}\{\chi_C : C \in \text{core}(\mathcal{C})\} = Q(b(\mathcal{C})) \cap \{x : x_u + x_v = 1, \{u, v\} \in E\}$.

Proof of Claim. (\subseteq) follows immediately from (\star) . (\supseteq) Pick a point x^* in the set on the right-hand side. Clearly, $x^* \in Q(b(\mathcal{C}))$. Since \mathcal{C} is ideal, so is $b(\mathcal{C})$, implying that for some $\lambda \in \mathbb{R}_+^{\mathcal{C}}$ with $\sum_{C \in \mathcal{C}} \lambda_C = 1$, we have that

$$x^* \geq \sum_{C \in \mathcal{C}} \lambda_C \chi_C.$$

Since for all $\{u, v\} \in E$, we have that $x_u^* + x_v^* = 1$ and $\{u, v\} \in b(\mathcal{C})$, equality must hold above and by (\star) , if $\lambda_C > 0$ then $C \in \text{core}(\mathcal{C})$. Hence, $x^* \in \text{conv}\{\chi_C : C \in \text{core}(\mathcal{C})\}$, as required. \diamond

For each $i \in [r]$, pick $u_i \in U_i$ and $v_i \in V_i$, and let \mathcal{C}' be the clutter over ground set $\{u_1, v_1, \dots, u_r, v_r\}$ obtained from $\text{core}(\mathcal{C})$ after contracting $V - \{u_1, v_1, \dots, u_r, v_r\}$. Notice that $|\{u_i, v_i\} \cap C| = 1$ for all $i \in [r]$ and $C \in \mathcal{C}'$. Observe that \mathcal{C}' is nothing but cuboid(S).¹

Claim 3. $\text{conv}\{\chi_C : C \in \mathcal{C}'\}$ can be described by $z \geq \mathbf{0}$, $z_{u_i} + z_{v_i} = 1, i \in [r]$, and

$$\sum_{B \cap U_i \neq \emptyset} |B \cap U_i| z_{u_i} + \sum_{B \cap V_j \neq \emptyset} |B \cap V_j| z_{v_j} \geq 1$$

for every $B \in b(\mathcal{C})$ such that for each $i \in [r]$, $B \cap (U_i \cup V_i) \subseteq U_i$ or V_i . Thus, (2) holds.

Proof of Claim. Observe that $\text{conv}\{\chi_C : C \in \mathcal{C}'\}$ is the projection of $\text{conv}\{\chi_C : C \in \text{core}(\mathcal{C})\}$ onto the coordinates $\{u_i, v_i : i \in [r]\}$. Thus, to give a description for $\text{conv}\{\chi_C : C \in \mathcal{C}'\}$, we may apply Fourier-Motzkin Elimination to the description of $\text{conv}\{\chi_C : C \in \text{core}(\mathcal{C})\}$ given by Claim 2, thereby giving us the claimed description. \diamond

Claim 4. $\text{conv}(S)$ is defined by $\mathbf{0} \leq x \leq \mathbf{1}$, and

$$\sum_{B \cap U_i \neq \emptyset} |B \cap U_i| x_i + \sum_{B \cap V_j \neq \emptyset} |B \cap V_j| (1 - x_j) \geq 1$$

for every minimal cover B of \mathcal{C} such that for each $i \in [r]$, $B \cap (U_i \cup V_i) \subseteq U_i$ or V_i . Thus, (3) holds.

Proof of Claim. This follows from Claim 3 by another application of Fourier-Motzkin Elimination. \diamond

Claim 4 finishes the proof. \square

¹Up to here, our proof is almost identical to the proof of Theorem 22, except that the statement of Claim 2 has been slightly revised to a more convenient formulation.

References

- [1] A. Abdi and G. Cornuéjols. A min-max theorem for clean tangled clutters. Submitted, 2022.
- [2] A. Abdi, G. Cornuéjols, N. Guričanová, and D. Lee. Cuboids, a class of clutters. *Journal of Combinatorial Theory, Series B*, 142:144 – 209, 2020.
- [3] A. Abdi, G. Cornuéjols, T. Huynh, and D. Lee. Idealness of k-wise intersecting families. *Mathematical Programming*, 2020.