Clean clutters and dyadic fractional packings

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Abstract

A vector is *dyadic* if each of its entries is a dyadic rational number, i.e., an integer multiple of $\frac{1}{2^k}$ for some nonnegative integer k. We prove that every clean clutter with a covering number of at least two has a dyadic fractional packing of value two. This result is best possible for there exist clean clutters with a covering number of three and no dyadic fractional packing of value three. Examples of clean clutters include ideal clutters, binary clutters, and clutters without an intersecting minor. Our proof is constructive and leads naturally to an albeit exponential algorithm. We improve the running time to quasi-polynomial in the *rank* of the input, and to polynomial in the binary case.

Keywords. Ideal clutter, cube-ideal set, dyadic fractional packing, quasi-polynomial time, Carathéodory's Theorem, projective geometries over $GF(2)$.

1 Introduction

Let $\mathcal{C}$ be a clutter over ground set V. The polyhedron $\left\{x \in \mathbb{R}^V_+ : x(C) \geq 1 \; \forall C \in \mathcal{C}\right\}$ is called the *set covering polyhedron associated with* $\mathcal{C}$, where $x(C) = \sum_{v\in C} x_v$. $\mathcal{C}$ is an *ideal clutter* if this polyhedron is integral [15]. Consider the following primal-dual pair of linear programs for an ideal clutter $\mathcal{C}$:

$$(P) \quad \begin{array}{ll} \min & \mathbf{1}^\top x \\ \text{s.t.} & x(C) \geq 1 \; \forall C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array} \qquad (D) \quad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum(y_C : v \in C, C \in \mathcal{C}) \leq 1 \; \forall v \in V \\ & y \geq \mathbf{0}. \end{array}$$

Basic polyhedral theory tells us that the primal (P) has an integral, in fact $0-1$, optimal solution. As a result, the linear program (P) computes the *covering number of* $\mathcal{C}$, that is, the minimum number of elements in V needed to intersect every member of $\mathcal{C}$.

Let us switch to the dual program. Any feasible solution y to (D) is called a *fractional packing of* $\mathcal{C}$ of *value* $\mathbf{1}^\top y$. For each element $v \in V$, the *congestion of* v in y is $\sum(y_C : v \in C, C \in \mathcal{C})$. Observe that a fractional packing assigns a nonnegative fraction to every member so that every element has congestion at most 1. Is there always an optimal fractional packing that is integral? Unfortunately, the answer is no. For example, the clutter $Q_6 := \{\{1,3,6\},\{1,4,5\},\{2,3,5\},\{2,4,6\}\}$ is ideal, but there is a unique optimal fractional packing assigning $\frac{1}{2}$, a fractional number, to every member [32]. Let us provide some context and then ask a better informed question.

A well-known and fundamental result in Combinatorial Optimization and Polyhedral Combinatorics is the theorem of Edmonds and Johnson stating that the clutter of (minimal) T-cuts of a graph is ideal [18]. A less known result is that of Lovász, stating that there is always an optimal fractional packing of T-cuts that is $\frac{1}{2}$-integral [27].

After applying tools from the theory of blocking clutters to the theorem of Edmonds and Johnson, we obtain that the clutter of (minimal) T-joins of a graph is also ideal [20] (see [13], §2). Seymour observed that within this class of ideal clutters, there are examples where (D) has no $\frac{1}{2}$-integral optimal solution [33]. However, motivated by conjectures of Tutte, Berge and Fulkerson, he conjectures that there should always exist an optimal fractional packing of T-joins that is $\frac{1}{4}$-integral [33] (see also [13], Conjecture 2.15). He made a case for this conjecture by showing that it would follow from the generalized Berge-Fulkerson conjecture on perfect matchings of r-graphs [33].

More generally, a vector is *dyadic* if it is $\frac{1}{2^k}$-integral for some integer $k \geq 0$. Inspired by the results, implications, and conjectures above, Seymour made the following conjecture about all ideal clutters:

Conjecture 1.1 (Seymour 1975, see [30], §79.3e)**.** *Every ideal clutter has an optimal fractional packing that is dyadic.*

In this paper, we prove that every ideal clutter with covering number at least two has a dyadic fractional packing of value two, thereby providing some evidence for Conjecture 1.1. Our proof uses the key insight that ideal clutters forbid two types of substructures: *deltas* which are clutters coming from projective planes, and *the blocker of an extended odd hole* which are clutters coming from odd holes in graphs.

A few words about Conjecture 1.1: We discuss this conjecture mainly in §7, where possible restrictions and extensions, as well as the binary case, are considered. In order to motivate the reader now, we point out that Conjecture 1.1 may be viewed as the set covering analogue of Fulkerson's theorem that every perfect set packing system is totally dual integral [19]. This conjecture has led the authors to introduce and study *totally dual dyadic systems* [2], and more generally *dyadic linear programming* [1]. Finally, Seymour has another conjecture that every ideal clutter should have an optimal fractional packing that is not only dyadic but $\frac{1}{4}$-integral (see [30], §79.3e). This conjecture, however, is studied in the more appropriate context of *k-wise intersecting families* [4, 5].

1.1 Clean clutters

Let us formally define clutters. Let V be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of V, called *members*. The family $\mathcal{C}$ is a *clutter* over *ground set* V if no member contains another one [17]. A subset $B \subseteq V$ is a *cover of* $\mathcal{C}$ if it intersects every member. The *covering number of* $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. The covering number of $\{\emptyset\}$ is assumed ∞, while the covering number of its blocker, $\{\}$, is 0.

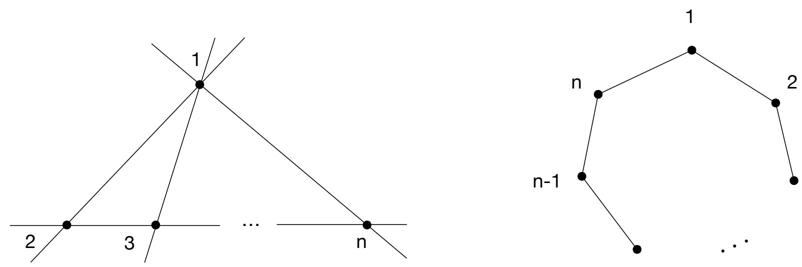

Figure 1: **Left**: The points and lines correspond to the elements and members of Δ_n. **Right**: The edges correspond to the minimum cardinality members of an extended odd hole with n elements.

A cover of $\mathcal{C}$ is *minimal* if it does not contain another cover. The family of the minimal covers of $\mathcal{C}$ forms another clutter over the same ground set, called the *blocker of* $\mathcal{C}$, and denoted $b(\mathcal{C})$ [17]. It can be readily checked that $b(b(\mathcal{C})) = \mathcal{C}$ [24, 17]. Lehman's width-length inequality, and also Fulkerson's theory of blocking polyhedra, implies that if a clutter is ideal, then so is its blocker [26, 20].

Take disjoint $I, J \subseteq V$. The *minor* of $\mathcal{C}$ obtained after *deleting* I and *contracting* J, denoted $\mathcal{C} \setminus I/J$, is the clutter over ground set $V - (I \cup J)$ whose members consist of the inclusion-wise minimal sets of $\{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$. The minor is *proper* if $I \cup J \neq \emptyset$. It can be readily checked that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [31]. In terms of the set covering polyhedron, deletion and contraction correspond to projection and restriction, respectively. As these operations preserve polyhedral integrality, we get that if a clutter is ideal, then so is every minor of it [32].

For an integer $n \geq 3$, denote by Δ_n the clutter over ground set $[n] := \{1, \ldots, n\}$ whose members are $\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}$ (see Figure 1). Observe that $b(\Delta_n) = \Delta_n$. Observe further that $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$ is a fractional vertex of the associated set covering polyhedron, so Δ_n is non-ideal. More generally, a *delta* is any clutter obtained from Δ_n after relabeling its ground set.

An *extended odd hole* is any clutter whose elements can be relabeled as $[n]$, for some odd integer $n \geq 5$, to obtain a clutter $\mathcal{C}$ such that $\mathcal{C} = \{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}\} \cup \mathcal{C}'$ where every member of $\mathcal{C}'$, if any, has cardinality at least 3 (see Figure 1). Observe that every cover of $\mathcal{C}$ has cardinality at least $\frac{n+1}{2}$. Observe further that $\frac{1}{2} \cdot \mathbf{1}$ is a fractional vertex of the associated set covering polyhedron. Thus an extended odd hole is non-ideal, implying in turn that the blocker of an extended odd hole is non-ideal.

Deltas, extended odd holes and their blockers are among the most basic classes of non-ideal clutters. In fact, with the sole exception of the lines of the Fano plane, these were the first non-ideal clutters found and studied by Lehman [26]. Thus, as a first step to studying idealness, it seems natural to study clutters without such minors. What is fascinating though is the recent discovery that finding the blocker of an extended odd hole as a minor is much easier than an extended odd hole. More precisely, it was proved that there is a polynomial time algorithm that, given a clutter, finds a delta or the blocker of an extended odd hole, or certifies that none exists ([6], Theorem 1.11). This stands in stark contrast with a result of Ding, Feng, and Zang that testing whether a clutter has a delta or an extended odd hole as a minor is NP-complete [16, 6]. In fact, this hardness result is the reason that testing

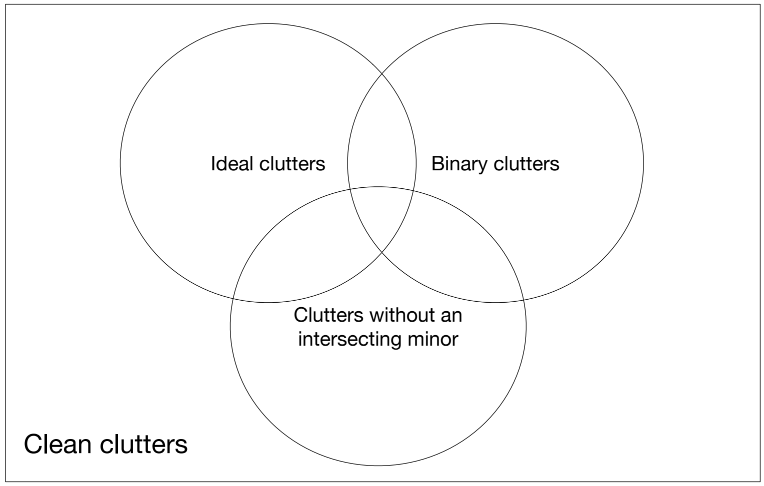

Figure 2: A Venn diagram of the world of clean clutters

idealness is co-NP-complete [16]. Inspired by these results, let us make the following definition:

Definition 1.2. *A clutter is* clean *if it has no minor that is a delta or the blocker of an extended odd hole.*

Notice that if a clutter is clean, then so is every minor of it. As was already pointed out, every ideal clutter is clean. However, unlike idealness, being clean is efficiently recognizable as shown by the result mentioned above [6]. Other than ideal clutters, there are two other important classes of clean clutters, as described below and depicted in Figure 2.

A clutter is *intersecting* if every two members intersect yet no element belongs to all members [6]. Notice that every delta and the blocker of every extended odd hole is an intersecting clutter. Thus clutters without an intersecting minor are also clean. A natural connection between clean clutters and this class was recently established in [6]. This class is important because it has been conjectured that idealness of such clutters is equivalent to the total dual integrality of the associated set covering linear system (P) [6]. This is in fact equivalent to the $\tau = 2$ *Conjecture* by Cornuéjols, Guenin and Margot [14].

$\mathcal{C}$ is a *binary* clutter if the symmetric difference of any odd number of members contains a member; equivalently, $\mathcal{C}$ is binary if $|C \cap B| \equiv 1 \pmod 2$ for all $C \in \mathcal{C}, B \in b(\mathcal{C})$ [25]. In particular, a clutter is binary if and only if its blocker is binary. For instance, the clutter of T-joins of a graph is binary. Observe that the deltas, extended odd holes and their blockers are not binary. If a clutter is binary, so is every minor of it [31]. Thus binary clutters are clean.

1.2 The existence of a dyadic fractional packing of value two

Consider a clean clutter with covering number at least two. To look for a dyadic fractional packing of value two in the clutter, we first move to a deletion minor that is minimal subject to having covering number at least two, find the desired fractional packing there, and then lift it back up to a desired fractional packing in the original clutter. Such a deletion minor conforms to the following definition:

Definition 1.3 ([4, 5]). *A clutter is* tangled *if its covering number is two, and every element belongs to a minimum cover.*

Observe that a clutter is tangled if, and only if, it has covering number at least two but every proper deletion minor has covering number less than two.

Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Denote by $G(\mathcal{C})$ the graph over vertex set V whose edges correspond to the minimal covers of $\mathcal{C}$ of cardinality two. The *rank of* $\mathcal{C}$, denoted $\mathrm{rank}(\mathcal{C})$, is the number of connected components of $G(\mathcal{C})$. We prove that every clean tangled clutter has a dyadic fractional packing of value two, one whose fractionality depends on the rank:

Theorem 1.4. *Let $\mathcal{C}$ be a clean tangled clutter of rank r. Then $\mathcal{C}$ has a $\frac{1}{2^{r-1}}$-integral packing of value two.*

In fact, motivated by Carathéodory's Theorem and also *projective geometries over the two-element field*, we conjecture that the denominator 2^{r-1} can be improved to 2^{k-1}, for some integer $k \geq 1$ such that $k \leq \log(r+1)$; see §3.3 and §7 for more details.[1] For now, let us present some consequences of Theorem 1.4.

Theorem 1.5. *Every clean clutter with covering number at least two has a dyadic fractional packing of value two.*

Proof of Theorem 1.5, assuming Theorem 1.4. Let $\mathcal{C}$ be a clean clutter with covering number at least two, and let $\mathcal{C}'$ be a deletion minor that is minimal subject to having covering number at least two. Then $\mathcal{C}'$ is necessarily a clean tangled clutter. It therefore follows from Theorem 1.4 that $\mathcal{C}'$, and therefore $\mathcal{C}$, has a dyadic fractional packing of value two. □

Theorem 1.5 is in a sense best possible, as there are clean clutters with covering number three and no dyadic fractional packing of value three – see §7.2 for examples.

Applying Theorem 1.5 to the special classes of clean clutters yields the following:

Corollary 1.6. *The following statements hold:*

(a) Every ideal clutter with covering number at least two has a dyadic fractional packing of value two.

(b) Every binary clutter with covering number at least two has a dyadic fractional packing of value two.

In particular, Corollary 1.6 (a) provides some evidence for Conjecture 1.1. For the joint special case of ideal binary clutters, Corollary 1.6 was strengthened very recently by using tools from Graph Theory and Matroid Theory: It is shown that every ideal binary clutter with covering number at least two has a $\frac{1}{4}$-integral packing of value two [5].

Finally, note that Theorem 1.5 also implies that every clutter without an intersecting minor and with covering number at least two, has a dyadic fractional packing of value two. However, this is a trivial statement as every such clutter has two disjoint members, by definition.

[1] All the log's in this paper are base 2.

1.3 Finding a dyadic fractional packing in quasi-polynomial time

To discuss the computational complexity of our algorithms in full generality, we assume that our clutters are inputted via an oracle. More precisely, a *filter oracle* for a clutter $\mathcal{C}$ consists of V along with an oracle which, given any set $X \subseteq V$, decides in unit time whether or not X contains a member [34].

We see two algorithms that, given a clean tangled clutter over ground set V, output a dyadic fractional packing $y^\star$ of value two. For both algorithms, the enumeration tree is binary, the height of the tree determines the fractionality of $y^\star$, the total number of leaves determines the support cardinality of $y^\star$, while the total number of nodes in the tree determines the number of calls to the filter oracle.

The first algorithm, called *Algorithm 1*, follows directly from the constructive proof of Theorem 1.4. Given that the clutter has rank r, the algorithm outputs a fractional packing of value two that is $\frac{1}{2^{r-1}}$-integral and whose support has cardinality at most 2^r. Moreover, the algorithm makes at most $(2^r - 1) \cdot |V|^2$ calls to the oracle and has running time $O(|V|^2)$ per call. While r can be much smaller than $|V|$, both the number of calls to the oracle and the support cardinality depend exponentially on r. See §5.2 for an explicit description of this algorithm, as well as its analysis.

We then see a revised algorithm, *Algorithm 2*, that improves the exponential dependence on r to quasi-polynomial:

Theorem 1.7. *Let $\mathcal{C}$ be a clean tangled clutter of rank r over ground set V inputted via a filter oracle. Then Algorithm 2 outputs a fractional packing $y^\star$ of value two that is $\frac{1}{2^{r-1}}$-integral and whose support has cardinality at most $r^{O(\log r)}$. Moreover, the algorithm makes $r^{O(\log r)} \cdot O(|V|^3)$ calls to the oracle and has running time $O(|V|^2)$ per call.*

In §6, we prove that Algorithm 2 has a linear dependence on r for binary clutters. More generally, we believe that the dependence on r can be further improved to a polynomial; see §7 for more details.

1.4 Outline of the paper

We prove Theorem 1.4 in §2. In that section, we also provide a key induction tool for clean tangled clutters. In §3, we apply Theorem 1.4 to *cuboids*, and see how Carathéodory's Theorem and projective geometries over $GF(2)$ come into play. After discussing some preliminaries on the computational complexity in §4, we present in §5 Algorithms 1 and 2 and the proof of Theorem 1.7. In §6, we prove that Algorithm 2 performs efficiently for binary clutters. In the final section, §7, we discuss potential improvements of Theorem 1.4 and Theorem 1.7, and debate possible extensions and restrictions of Conjecture 1.1.

2 Existence of a dyadic fractional packing

In this section, we present two preliminaries on clean tangled clutters in §2.1, prove an important inductive tool in §2.2, and then in §2.3, we prove Theorem 1.4 on the existence of dyadic fractional packings of value two.

2.1 Preliminaries

Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Recall that $G(\mathcal{C})$ denotes the graph over vertex set V whose edges correspond to the minimal covers of cardinality two. We need the following preliminaries from [11]:

Remark 2.1 ([11]). *Let $\mathcal{C}$ be a clean tangled clutter. Then $G(\mathcal{C})$ is bipartite.*

Proof. Suppose otherwise. Let V be the ground set of $\mathcal{C}$. As $G(\mathcal{C})$ is non-bipartite, it has a chordless odd cycle; let us label its vertices as $u_1, \ldots, u_n \in V$ for some odd integer $n \geq 3$. Let $J := V - \{u_1, \ldots, u_n\}$. Observe that $b(\mathcal{C}) \setminus J$ is either Δ_3 or an extended odd hole, implying in turn that $\mathcal{C}/J$ is either $b(\Delta_3) = \Delta_3$ or the blocker of an extended odd hole, a contradiction as $\mathcal{C}$ is clean. $\square$

Theorem 2.2 ([11]). *Let $\mathcal{C}$ be a clean tangled clutter, where $G(\mathcal{C})$ is connected and has bipartition $\{U, U'\}$. Then neither U nor U' is a cover.*

We also need the following remark:

Remark 2.3. *Let $\mathcal{C}$ be a clean tangled clutter, let $G := G(\mathcal{C})$, and let $\{U, U'\}$ be the bipartition of a connected component of G. Then every member of $\mathcal{C}$ disjoint from U contains U'.*

Proof. Every edge of G gives a cover of $\mathcal{C}$, so every member of $\mathcal{C}$ is a vertex cover of G, implying in turn that every member of $\mathcal{C}$ disjoint from U must contain U'. $\square$

Theorem 2.2 and Remark 2.3 have the following immediate consequence:

Corollary 2.4. *Let $\mathcal{C}$ be a clean tangled clutter, where $G := G(\mathcal{C})$ is connected and has bipartition $\{U, U'\}$. Then $U, U' \in \mathcal{C}$ (so $\mathcal{C}$ has two disjoint members).*

Proof. By Theorem 2.2, U is not a cover, so there is a member C such that $C \cap U = \emptyset$, implying that $C \subseteq U'$. However, any such member has to contain U' by Remark 2.3, so $U' \subseteq C$, implying in turn that $C = U'$, so $U' \in \mathcal{C}$. Similarly, $U \in \mathcal{C}$, as required. $\square$

2.2 An inductive tool for clean tangled clutters

Given a graph $G = (V, E)$ and a subset $X \subseteq V$, denote by $G[X]$ the subgraph induced on vertices X. We are now ready to prove the following important result, which is crucial for the proof of Theorem 1.4.

Theorem 2.5. *Let $\mathcal{C}$ be a clean tangled clutter. Then $G(\mathcal{C})$ is a bipartite graph where every vertex is incident with an edge. Moreover, if $G(\mathcal{C})$ is not connected and $\{U, U'\}$ is the bipartition of a connected component, then $\mathcal{C} \setminus U/U'$ is a clean tangled clutter.*

Proof. Let $\mathcal{C}$ be a clean tangled clutter over ground set V, and let $G := G(\mathcal{C})$. Then by Remark 2.1 and the tangled property, G is a bipartite graph where every vertex is incident with an edge.

Let $\{U, U'\}$ be the bipartition of a connected component of G. Clearly $\mathcal{C} \setminus U/U'$ is clean and every element of it appears in a cardinality-two cover. To prove that $\mathcal{C} \setminus U/U'$ is tangled, it remains to prove that $\tau(\mathcal{C} \setminus U/U') \geq 2$.

Suppose for a contradiction that $\tau(\mathcal{C} \setminus U/U') \leq 1$. Then there exists a $B \in b(\mathcal{C})$ such that $B \cap U' = \emptyset$ and $|B - U| \leq 1$. Let $\mathcal{C}'$ be the minor of $\mathcal{C}$ obtained after deleting $B - U$ and contracting $V - (U \cup U' \cup B)$. Notice that $\mathcal{C}'$ has ground set $U \cup U'$. Clearly, $\mathcal{C}'$ is a clean clutter.

We claim that $\tau(\mathcal{C}') \geq 2$. If $B - U = \emptyset$, then $\tau(\mathcal{C}') \geq \tau(\mathcal{C}) = 2$. Otherwise, $|B - U| = 1$. In this case, a cover of cardinality one in $\mathcal{C}'$ would come from a cardinality-two cover in $\mathcal{C}$ contained in $U \cup U' \cup (B - U)$ and with an element in $B - U$, that is, it would come from an edge of G with an end in $U \cup U'$ and another end in $B - U$. However, as $G[U \cup U']$ is a connected component of G, G has no edge with an end in $U \cup U'$ and another in $B - U$. As a result, $\mathcal{C}'$ has no cover of cardinality one, so $\tau(\mathcal{C}') \geq 2$, as claimed.

Since every cardinality-two cover of $\mathcal{C}$ contained in $U \cup U'$ is also a cover of $\mathcal{C}'$, the latter must be a tangled clutter. Let $G' := G(\mathcal{C}')$. Then G' is a bipartite graph by Remark 2.1. As $G[U \cup U'] \subseteq G'$, G' is connected and its bipartition is inevitably $\{U, U'\}$. It therefore follows from Corollary 2.4 that U, U' are both members of $\mathcal{C}'$. However, $B \cap U = B \cap (U \cup U') \in b(\mathcal{C}')$ is disjoint from $U' \in \mathcal{C}'$, a contradiction. □

2.3 The proof of Theorem 1.4

We need the following two lemmas:

Lemma 2.6. *Let $\mathcal{C}$ be a clean tangled clutter, where $G(\mathcal{C})$ is not connected. Let $\{U, U'\}$ be the bipartition of a connected component of $G(\mathcal{C})$, and let z, z' be fractional packings of $\mathcal{C} \setminus U/U', \mathcal{C}/U \setminus U'$ of value two, respectively. Let $y, y' \in \mathbb{R}^{\mathcal{C}}_{+}$ be defined as follows:* [2]

$$y_C := \begin{cases} z_{C-U'} & \text{if } C \cap U = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y'_C := \begin{cases} z'_{C-U} & \text{if } C \cap U' = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then $\frac{1}{2}y + \frac{1}{2}y'$ is a fractional packing of $\mathcal{C}$ of value two. Moreover, if z, z' are dyadic vectors, then so is $\frac{1}{2}y + \frac{1}{2}y'$.

Proof. We leave this as an exercise for the reader. □

Given a clean tangled clutter $\mathcal{C}$, recall that $\mathrm{rank}(\mathcal{C})$ denotes the number of connected components of the bipartite graph $G(\mathcal{C})$.

Lemma 2.7. *Let $\mathcal{C}$ be a clean tangled clutter, where $G := G(\mathcal{C})$ is not connected, and let $\{U, U'\}$ be the bipartition of a connected component of G. Then $\mathrm{rank}(\mathcal{C} \setminus U/U') \leq \mathrm{rank}(\mathcal{C}) - 1$.*

Proof. Let $r := \mathrm{rank}(\mathcal{C})$, and let $\{U_i, V_i\}, i \in [r-1]$ be the bipartitions of the connected components of G different from $G[U \cup U']$. Let $\mathcal{C}' := \mathcal{C} \setminus U/U'$, which is clean and tangled by Theorem 2.5, and let $G' := G(\mathcal{C}')$. As $G[U_i \cup V_i] \subseteq G'$ for all $i \in [r-1]$, G' has at most $r - 1$ connected components, so $\mathrm{rank}(\mathcal{C}') \leq r - 1$. □

[2] Notice that by Remark 2.3, if $C \cap U = \emptyset$ then $U' \subseteq C$ so $C - U'$ is a member of $\mathcal{C} \setminus U/U'$, and if $C \cap U' = \emptyset$ then $U \subseteq C$ so $C - U$ is a member of $\mathcal{C}/U \setminus U'$.

We are now ready to prove Theorem 1.4:

Proof of Theorem 1.4. Let $\mathcal{C}$ be a clean tangled clutter over ground set V and of rank r. We prove, by induction on $r \geq 1$, that $\mathcal{C}$ has a $\frac{1}{2^{r-1}}$-integral packing of value two. Let $G := G(\mathcal{C})$, a bipartite graph where every vertex is incident with an edge by Theorem 2.5, and let $\{U, U'\}$ be the bipartition of a connected component of G. For the base case $r = 1$, notice that $U, U' \in \mathcal{C}$ by Corollary 2.4, so $\{U, U'\}$ gives a $\frac{1}{2^{r-1}}$-integral packing of value two, thereby proving the base case. For the induction step, assume that $r > 1$, that is, G is not connected. Then $\mathcal{C} \setminus U/U'$ and $\mathcal{C}/U \setminus U'$ are clean and tangled by Theorem 2.5. Let $r_1 := \mathrm{rank}(\mathcal{C} \setminus U/U')$ and $r_2 := \mathrm{rank}(\mathcal{C}/U \setminus U')$. By Lemma 2.7, $r_1 \leq r-1$ and $r_2 \leq r-1$. Hence, by the induction hypothesis, there exist fractional packings z, z' of $\mathcal{C} \setminus U/U', \mathcal{C}/U \setminus U'$ of value two that are $\frac{1}{2^{r_1-1}}$-integral and $\frac{1}{2^{r_2-1}}$-integral, respectively. Let $y, y' \in \mathbb{R}_+^{\mathcal{C}}$ be defined as follows:

$$y_C := \begin{cases} z_{C-U'} & \text{if } C \cap U = \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \quad y'_C := \begin{cases} z'_{C-U} & \text{if } C \cap U' = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then $\frac{1}{2}y + \frac{1}{2}y'$ is a fractional packing of $\mathcal{C}$ of value two by Lemma 2.6, one that is $\frac{1}{2^{k-1}}$-integral for $k = 1 + \max\{r_1, r_2\}$. As $k \leq r$, $\frac{1}{2}y + \frac{1}{2}y'$ is the desired dyadic fractional packing, thereby completing the induction step. ☐

3 Carathéodory's Theorem and projective geometries

The reader may wonder about the significance of Theorem 1.4, as it only guarantees dyadic fractional packings of value two, and none of higher value. The theory of *cuboids* addresses this concern, by which we derive an insightful consequence of Theorem 1.4. Let us elaborate.

Take an integer $n \geq 1$ and a subset $S \subseteq \{0,1\}^n$. The *cuboid of* S, denoted $\mathrm{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members have incidence vectors $\{(p_1, 1-p_1, \ldots, p_n, 1-p_n) : p \in S\}$. Observe that $\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}$ are covers of $\mathrm{cuboid}(S)$. As a consequence, if the points in S do not agree on a coordinate, then $\mathrm{cuboid}(S)$ has covering number two, so it is a tangled clutter. In the case where the cuboid is a clean clutter, too, we can apply Theorem 1.4 and obtain the following consequence:

Theorem 3.1. *Take an integer $n \geq 1$ and a subset $S \subseteq \{0,1\}^n$ whose points do not agree on a coordinate and whose cuboid is a clean clutter. Then $\frac{1}{2} \cdot \mathbf{1}$ can be written as a $\frac{1}{2^n}$-integral convex combination of the points in S.*

Proof. Let $\mathcal{C} := \mathrm{cuboid}(S)$. We know that $\mathcal{C}$ is a clean tangled clutter, with rank at most n as $G(\mathcal{C})$ contains edges $\{2i-1, 2i\}, i \in [n]$. It therefore follows from Theorem 1.4 that $\mathcal{C}$ has a $\frac{1}{2^{n-1}}$-integral packing $y \in \mathbb{R}_+^{\mathcal{C}}$ of value two. For each $i \in [n]$, every member of $\mathcal{C}$ contains exactly one of $2i-1, 2i$, so

$$2 = \mathbf{1}^\top y = \sum (y_C : C \in \mathcal{C}, 2i-1 \in C) + \sum (y_C : C \in \mathcal{C}, 2i \in C) \leq 1 + 1 = 2,$$

where the inequality follows from the fact that every element has congestion at most 1 in y. Equality must hold throughout above, so

$$\sum (y_C : C \in \mathcal{C}, 2i-1 \in C) = \sum (y_C : C \in \mathcal{C}, 2i \in C) = 1 \quad \forall i \in [n].$$

For each point $p \in S$ with corresponding member $C \in \mathcal{C}$, define $z_p := \frac{1}{2} \cdot y_C$. The equality above implies that

$$\sum_{p \in S} z_p \cdot p = \frac{1}{2} \cdot \mathbf{1}.$$

As $\mathbf{1}^\top z = \frac{1}{2} \cdot \mathbf{1}^\top y = 1$, the equality above expresses $\frac{1}{2} \cdot \mathbf{1}$ as a $\frac{1}{2^n}$-integral convex combination of the points in S, as required. □

The assumption in Theorem 3.1 that the cuboid is clean may seem too unfamiliar to the reader, so let us focus on two insightful special cases, namely ideal cuboids in §3.1 and binary cuboids in §3.2. Then we pose a conjecture in §3.3 that would strengthen Theorem 3.1.

3.1 Cube-ideal sets

Take an integer $n \geq 1$ and a subset $S \subseteq \{0,1\}^n$. We say that S is *cube-ideal* if every facet of $\mathrm{conv}(S)$ is either $x_i \geq 0$ or $x_i \leq 1$ for some $i \in [n]$, or it is a *generalized set covering inequality*:

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \quad I, J \subseteq [n], I \cap J = \emptyset.$$

Generalized set covering inequalities were studied previously by Hooker [23], Guenin [22], and Nobili and Sassano [28]. The notion of cube-idealness was defined in [12] and studied in [3].

Let S be a cube-ideal set whose points do not agree on a coordinate. Then every valid generalized set covering inequality for $\mathrm{conv}(S)$ must consist of at least two variables. Subsequently, $\frac{1}{2} \cdot \mathbf{1} \in \mathrm{conv}(S)$. In fact, a set is cube-ideal if and only if its cuboid is an ideal clutter ([3], Theorem 1.6). We may therefore apply Theorem 3.1 to derive a stronger conclusion:

Corollary 3.2. *Take an integer $n \geq 1$ and a cube-ideal set $S \subseteq \{0,1\}^n$ whose points do not agree on a coordinate. Then $\frac{1}{2} \cdot \mathbf{1}$ can be written as a $\frac{1}{2^n}$-integral convex combination of the points in S.*

3.2 Binary spaces and projective geometries

Take an integer $n \geq 1$ and a subset $S \subseteq \{0,1\}^n$. The set S is a *vector space over $GF(2)$*, or simply a *binary space*, if $a \triangle b \in S$ for all $a, b \in S$. In particular, a nonempty binary space necessarily contains $\mathbf{0}$. It can be readily checked that a set is a binary space if and only if its cuboid is a binary clutter [5]. We may therefore apply Theorem 3.1 to derive the following consequence:

Corollary 3.3. *Take an integer $n \geq 1$ and a binary space $S \subseteq \{0,1\}^n$ whose points do not agree on a coordinate. Then $\frac{1}{2} \cdot \mathbf{1}$ can be written as a $\frac{1}{2^n}$-integral convex combination of the points in S.*

Alternative proof. Let S be a binary space whose points do not agree on a coordinate. Basic Linear Algebra tells us that $|S| = 2^r$ where r is the $GF(2)$-rank of S, and that for each $i \in [n]$, $|S \cap \{x : x_i = 0\}| = |S \cap \{x : x_i = 1\}| = 2^{r-1}$. As a result, the point $\frac{1}{2} \cdot \mathbf{1}$ can be expressed as the uniform convex combination $\frac{1}{2^r} \cdot \mathbf{1} \in \mathbb{R}^S_+$, thereby proving Corollary 3.3. □

Let us describe an important class of binary spaces. Take an integer n of the form $n = 2^k - 1$ for some integer $k \geq 1$. Let A be the $k \times n$ matrix whose columns are all the $0-1$ vectors of dimension k that are nonzero. Let $\mathrm{cocycle}(PG(k-1,2)) \subseteq \{0,1\}^n$ be the row space of A over $GF(2)$. Observe that $\mathrm{cocycle}(PG(k-1,2))$ is a binary space with $n+1$ points, which do not agree on a coordinate. In fact,

Theorem 3.4 ([8])**.** *For every integer $k \geq 1$ and $n := 2^k - 1$, the convex hull of $\mathrm{cocycle}(PG(k-1,2))$ is an n-dimensional simplex. Moreover, $\frac{1}{2} \cdot \mathbf{1}$ is expressed as a unique convex combination of the points in S, which has support cardinality $n+1$, and where every coefficient is equal to $\frac{1}{2^k}$.*

As suggested by our notation, $\mathrm{cocycle}(PG(k-1,2))$ is the cocycle space of some binary matroid, called the *rank-k projective geometry over $GF(2)$* and denoted $PG(k-1,2)$. The rank-1 projective geometry is the graphic matroid of a triangle, whereas the rank-2 projective geometry is known as the *Fano matroid*.[3][4]

3.3 Carathéodory's Theorem and a conjecture

Let $S \subseteq \{0,1\}^n$ be a subset whose points do not agree on a coordinate and whose cuboid is a clean clutter. By Theorem 3.1, $\frac{1}{2} \cdot \mathbf{1}$ belongs to the convex hull of S. We can now apply Carathéodory's Theorem to write $\frac{1}{2} \cdot \mathbf{1}$ as the convex hull of S with at most $n+1$ nonzero coefficients. A natural question is whether this upper bound on the number of nonzero coefficients is tight for our setting, and indeed, Theorem 3.4 from the previous subsection demonstrates that the cocycle space of every projective geometry meets the upper bound of $n+1$. The next natural question is whether one can achieve the upper bound by a dyadic convex combination:

Conjecture 3.5. *Take an integer $n \geq 1$ and a subset $S \subseteq \{0,1\}^n$ whose points do not agree on a coordinate and whose cuboid is a clean clutter. Then $\frac{1}{2} \cdot \mathbf{1}$ can be written as a $\frac{1}{2^k}$-integral convex combination of the points in S, where k is an integer such that $1 \leq k \leq \log(n+1)$.*

Notice that the upper bound given on k is tight in light of Theorem 3.4. This conjecture is a special case of a more general conjecture in §7 (namely, Conjecture 7.1).

4 Preliminaries for computational complexity

In §4.1, we discuss the filter oracle in some detail, and in §4.2, we discuss enumeration trees of exponential, quasi-polynomial, and polynomial sizes.

4.1 The filter oracle

Let $\mathcal{C}$ be a clutter over ground set V. Recall that a filter oracle for $\mathcal{C}$ consists of V along with an oracle which, given any set $X \subseteq V$, decides in unit time whether or not X contains a member. Observe that,

[3] In the context of clutters, rank refers to the $\mathbb{R}$-rank, while in the context of binary matroids, rank refers to the $GF(2)$-rank.

[4] We warn the reader of the overuse of the term "rank"; this term has a different meaning here than in the rest of the paper, and in order to avoid further confusion, we reserve the term to refer to the number of connected components of $G(\mathcal{C})$ for a clean tangled clutter $\mathcal{C}$.

Remark 4.1. *Given a filter oracle for a clutter $\mathcal{C}$ over ground set V, one can produce all covers of cardinality two with $\binom{|V|}{2}$ calls to the oracle.*

Proof. Given $X \subseteq V$, X is a cover if and only if $V - X$ does not contain a member of $\mathcal{C}$, so we can test in unit time whether or not X is a cover. As a result, one can produce all covers of cardinality at most two by querying the oracle accordingly for each set in $\{X \subseteq V : |X| = 2\}$. □

Remark 4.2 ([34])**.** *Given a filter oracle for a clutter $\mathcal{C}$ over ground set V, and given disjoint $I, J \subseteq V$, one has a filter oracle for $\mathcal{C} \setminus I/J$.*

Proof. Given $X \subseteq V - (I \cup J)$, X contains a member of $\mathcal{C} \setminus I/J$ if and only if $X \cup J$ contains a member of $\mathcal{C}$, so we can test in unit time whether or not X contains a member of $\mathcal{C} \setminus I/J$. □

Remark 4.3 ([34])**.** *Given a filter oracle for a clutter $\mathcal{C}$ over ground set V, one has a filter oracle for $b(\mathcal{C})$.*

Proof. Given $X \subseteq V$, X contains a member of $b(\mathcal{C})$ if and only if $V - X$ does not contain a member of $\mathcal{C}$, so we can test in unit time whether or not X contains a member of $b(\mathcal{C})$. □

4.2 Binary trees

The enumeration trees of Algorithms 1 and 2 are rooted binary trees where every node is labeled. Here we analyze the height, the number of leaves, and the number of nodes of such trees. We need the following lemma:

Lemma 4.4. *Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function such that $f(1) = 1$, and*

$$f(x) \leq 1 + f(x-1) + f(\lfloor 3x/4 \rfloor) \quad \forall\, x \geq 2.$$

Then, for each $x \geq 2$, the following statements hold:

(a) $f(x) \leq (1 + 2x - 2\lfloor 3x/4 \rfloor) \cdot f(\lfloor 3x/4 \rfloor)$

(b) $f(x) \leq (1 + 2x - 2\lfloor 3x/4 \rfloor)^{1+\frac{\log x}{\log 4/3}} \cdot f(1) \leq (1 + 2x - 2\lfloor 3x/4 \rfloor)^{1+2.41 \log x}$

In particular, $f(x) = x^{O(\log x)}$.

Proof. **(a)** A simple inductive argument, together with the fact that f is non-decreasing, tells us that for every integer $k \in [x-1]$,

$$f(x) \leq k + f(x-k) + k \cdot f(\lfloor 3x/4 \rfloor).$$

Letting $k^\star := x - \lfloor 3x/4 \rfloor$ we get

$$f(x) \leq k^\star + (k^\star + 1) \cdot f(\lfloor 3x/4 \rfloor) \leq (2k^\star + 1) \cdot f(\lfloor 3x/4 \rfloor)$$

thereby proving (a). **(b)** It follows from (a) and the monotonicity of f that

$$f(x) \leq (2k^\star + 1)^\ell f(\lfloor (3/4)^\ell x \rfloor).$$

Choosing $\ell = \lceil \frac{\log x}{\log 4/3} \rceil$ yields (b). □

We are now equipped to prove the following:

Theorem 4.5. *Let T be a binary tree, where every node is labeled with an integer in $\mathbb{N}$, called the* rank *of the node. Suppose that the root has rank r, and the rank of each child is at least one less than the rank of the parent. Then the following statements hold:*

(a) The tree has height at most $r-1$, the number of leaves is at most 2^{r-1}, while the number of nodes is at most $2^r - 1$.

(b) Assume that if a non-leaf node has rank x, then the sum of the ranks of its two children is at most $\frac{3}{2}(x-1)$. Then the tree has height at most $r-1$, the number of leaves is $r^{O(\log r)}$, and the number of nodes is also $r^{O(\log r)}$.

(c) Assume that if a non-leaf node has rank x, then each of its two children has rank at most $\frac{1}{2}(x-1)$. Then the tree has height at most $\log(r+1)-1$, the number of leaves is at most $\frac{r+1}{2}$, and the number of nodes is at most r.

Proof. **(a)** We leave this as an exercise for the reader. **(b)** It is clear that the tree has height at most $r-1$. For each node of rank x, denote by $f(x)$ the maximum possible number of descendants of the node, including the node itself. Clearly, $f:\mathbb{N}\to\mathbb{N}$ is a non-decreasing function such that $f(1)=1$. Moreover, our hypothesis implies that for each $x\geq 2$, $f(x)\leq 1+f(y)+f(z)$ for some integers $y,z\in[x-1]$ such that $y+z\leq\frac{3}{2}(x-1)$. Since f is non-decreasing, it follows that $f(x)\leq 1+f(x-1)+f(\lfloor 3x/4\rfloor)$. We may therefore apply Lemma 4.4 to conclude that $f(r)=r^{O(\log r)}$. Thus, the number of nodes in the tree is $r^{O(\log r)}$, implying in turn that the number of leaves is $r^{O(\log r)}$. **(c)** We leave this as an exercise for the reader. □

5 Finding a dyadic fractional packing of value two

In this section, we present the two promised algorithms for finding a dyadic fractional packing of value two in a clean tangled clutter. Both algorithms rely on the notion of a *reduction*, introduced in §5.1, which is a bi-set that reduces the problem of finding the desired fractional packing in a clean tangled clutter to that of two smaller clean tangled clutters. Algorithm 1 and its analysis are then presented in §5.2. After refining the notion of a reduction to that of an *incrementally maximal* reduction in §5.3, we present Algorithm 2 and the proof of Theorem 1.7 in §5.4.

5.1 Reductions

Let $\mathcal{C}$ be a clean tangled clutter over ground set V, and let $G:=G(\mathcal{C})$. A *reduction* is a set $\{I,J\}$ such that

(R1) $I,J\subseteq V$ are nonempty disjoint subsets, $G[I\cup J]$ is the union of some connected components of G, and I,J is a (proper) bicoloring of $G[I\cup J]$, and

(R2) $\tau(\mathcal{C}\setminus I/J)\geq 2$ and $\tau(\mathcal{C}/I\setminus J)\geq 2$.

Observe that every reduction $\{I,J\}$ must also satisfy the following two properties:

(R3) Every member of $\mathcal{C}$ disjoint from I contains J, and every member of $\mathcal{C}$ disjoint from J contains I: This follows from (R1) with an argument similar to that of Remark 2.3.

(R4) $\text{rank}(\mathcal{C}\setminus I/J) < \text{rank}(\mathcal{C})$ and $\text{rank}(\mathcal{C}/I\setminus J) < \text{rank}(\mathcal{C})$: This follows from (R1) and (R2) with an argument similar to that of Lemma 2.7.

A reduction $\{I,J\}$ is *proper* if $I\cup J\neq V$. We point out that a reduction may indeed be non-proper. In fact, we have the following characterization:

Remark 5.1. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V, and let $\{I,J\}$ be a reduction. Then the following statements hold:*

(R2a) if $\{I,J\}$ is a proper reduction, then $\mathcal{C}\setminus I/J, \mathcal{C}/I\setminus J$ are clean tangled clutters,

(R2b) $\{I,J\}$ is a non-proper reduction if, and only if, I,J are members of $\mathcal{C}$.

Proof. **(R2a)** can be readily checked. **(R2b)** ($\Rightarrow$) As $\mathcal{C}\setminus I/J$ is tangled, I is not a cover, so $V-I=J$ must contain a member. It now follows from (R3) that J is a member. As $\mathcal{C}/I\setminus J$ is also tangled, it follows similarly that I is also a member. ($\Leftarrow$) Observe that I,J are members that are in fact disjoint, because $\{I,J\}$ is a reduction. Since $\mathcal{C}$ is tangled, it follows that $I\cup J=V$, implying in turn that $\{I,J\}$ is a non-proper reduction. □

A proper reduction allows us to reduce the problem of finding a dyadic fractional packing of value two in a clean tangled clutter to that of two smaller clean tangled clutters, as outlined in the following extension of Lemma 2.6:

Lemma 5.2. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Suppose $\{I,J\}$ is a proper reduction. Let z,z' be fractional packings of $\mathcal{C}\setminus I/J, \mathcal{C}/I\setminus J$ of value two, respectively. Let $y,y'\in\mathbb{R}^{\mathcal{C}}_+$ be defined as follows:*

$$y_C := \begin{cases} z_{C-J} & \text{if } C\cap I=\emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y'_C := \begin{cases} z'_{C-I} & \text{if } C\cap J=\emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then the following statements hold:

(a) The assignment $y^\star := \frac{1}{2}y+\frac{1}{2}y'$ is a fractional packing of $\mathcal{C}$ of value two, and $|\text{support}(y^\star)|\leq|\text{support}(z)|+|\text{support}(z')|$.

(b) If I,J are explicitly provided, and the nonzero entries of z,z' are explicitly provided, then the nonzero entries of $y^\star$ can be explicitly derived with at most $|\text{support}(z)|+|\text{support}(z')|$ pairwise additions.

(c) Assume that $\mathcal{C}$ is inputted via a filter oracle. If I,J are found with a calls to the oracle with running time t_a per call, z is found with b calls to the oracle with running time t_b per call, and z' is found with c calls to the oracle with running time t_c per call, then $y^\star$ is found with $a+b+c$ calls to the oracle with running time $\max\{t_a,t_b,t_c\}$ per call.

Proof. **(a)** can be readily checked. **(b)** follows upon observing that the nonzero entries of y, y' can be found as follows, by (R3):

- For every nonzero entry z_A of z, we get a corresponding nonzero entry $y_{A\cup J}$ of y.
- For every nonzero entry z'_B of z', we get a corresponding nonzero entry $y'_{B\cup I}$ of y'.

(c) is straightforward. □

5.2 Algorithm 1

Let $\mathcal{C}$ be a clean tangled clutter, and let $\{U, U'\}$ be the bipartition of a connected component of $G(\mathcal{C})$. If $G(\mathcal{C})$ is connected, then $\{U, U'\}$ yields an integral packing of value two by Corollary 2.4. Otherwise, it follows from Theorem 2.5 that $\{U, U'\}$ is a proper reduction. Such singleton reductions have led to the proof of Theorem 1.4. The underlying algorithm is presented explicitly in Algorithm 1. Let us use Lemma 5.2 to analyze the algorithm.

Algorithm 1: A basic algorithm for finding a dyadic fractional packing of value two

Input: A filter oracle for a clean tangled clutter $\mathcal{C}$

Output: A dyadic fractional packing y of value two, provided explicitly

Initialize: Let $G := G(\mathcal{C})$, $r := \operatorname{rank}(\mathcal{C})$, and $\{U, U'\}$ the bipartition of a connected component of G;

if $r = 1$ **then**

  Let $y \in \mathbb{R}^{\mathcal{C}}_+$ be the incidence vector of the packing U, U';

  Output y.

else

  Let $\mathcal{C}_1 := \mathcal{C} \setminus U/U'$ and $\mathcal{C}_2 := \mathcal{C}/U \setminus U'$;

  Run Algorithm 1 on $\mathcal{C}_1$ (the filter oracle for $\mathcal{C}$ gives one for $\mathcal{C}_1$ by Remark 4.2) with output z ;

  Run Algorithm 1 on $\mathcal{C}_2$ (the filter oracle for $\mathcal{C}$ gives one for $\mathcal{C}_2$ by Remark 4.2) with output z';

  Define y as in Lemma 5.2;

  Output y.

Let us analyze Algorithm 1:

Theorem 5.3. *Let $\mathcal{C}$ be a clean tangled clutter of rank r over ground set V inputted via a filter oracle. Then Algorithm 1 outputs a fractional packing $y^\star$ of value two that is $\frac{1}{2^{r-1}}$-integral and whose support has cardinality at most 2^r. Moreover, the algorithm makes at most $(2^r - 1) \cdot |V|^2$ calls to the oracle and has running time $O(|V|^2)$ per call.*

Proof. Denote by T the enumeration tree of Algorithm 1. Then T is a rooted binary tree where each node is labeled with the rank of the corresponding clutter. Observe that the root has rank r, the leaves are the rank 1 nodes, and the rank of each child is at least one less than the rank of the parent by (R4). It therefore follows from Theorem 4.5 (a) that T has height $\le r - 1$, its number of leaves is $\le 2^{r-1}$, while the total number of nodes is $\le 2^r - 1$.

As the tree has height $\leq r-1$ and the number of leaves of the tree is at most 2^{r-1}, it follows from Lemma 5.2 that $y^\star$ is $\frac{1}{2^{r-1}}$-integral and its support has cardinality $\leq 2 \times 2^{r-1} = 2^r$.

The initialization step of the algorithm takes at most $|V|^2$ calls to the oracle with running time $O(|V|^2)$ per call. More precisely, it takes at most $|V|^2$ calls to the oracle to build $G(\mathcal{C})$ by Remark 4.1, and after the last call, it takes running time $O(|V|^2)$ to find r and $\{U, U'\}$. Thus, for each node of the tree T, at most $|V|^2$ calls to the oracle are made. Since the number of nodes of the tree is $\leq 2^r - 1$, the algorithm makes at most $(2^r - 1) \cdot |V|^2$ calls to the oracle and has running time $O(|V|^2)$ per call. □

5.3 Incrementally maximal reductions

Let $\mathcal{C}$ be a clean tangled clutter over ground set V, and let $G := G(\mathcal{C})$. A reduction $\{I, J\}$ is *incrementally non-maximal* if there exists another reduction $\{I', J'\}$ such that $I \subsetneq I'$ (and thus $J \subsetneq J'$), and $G[(I' \cup J') - (I \cup J)]$ consists of exactly one connected component of G. A reduction is *incrementally maximal* if it is not incrementally non-maximal.

We modify Algorithm 1 by replacing singleton reductions by incrementally maximal reductions at every iteration of the algorithm. We prove that this simple change drastically improves the number of calls to the oracle as well as the support cardinality of the output. This improvement is due to the fact that for an incrementally maximal reduction $\{I, J\}$ that is proper, we have

$$\mathrm{rank}(\mathcal{C} \setminus I/J) + \mathrm{rank}(\mathcal{C}/I \setminus J) \leq \frac{3}{2}(\mathrm{rank}(\mathcal{C}) - 1).$$

Let us prove this inequality. We need a few ingredients.

Lemma 5.4. *Let $\mathcal{C}$ be a clean tangled clutter, where $G(\mathcal{C})$ has exactly two connected components with bipartitions $\{U, U'\}, \{W, W'\}$. Then $U \cup W, U \cup W', U' \cup W, U' \cup W' \in \mathcal{C}$.*

Proof. Denote by V the ground set of $\mathcal{C}$, and let $G := G(\mathcal{C})$. Consider the minor $\mathcal{C}' := \mathcal{C} \setminus U/U'$. Then $\mathcal{C}'$ is a clean tangled clutter by Theorem 2.5. Clearly $G[W \cup W'] \subseteq G(\mathcal{C}')$, so $G(\mathcal{C}')$ is a connected, bipartite graph whose bipartition is inevitably $\{W, W'\}$. It therefore follows from Corollary 2.4 that $W, W' \in \mathcal{C}'$, implying in turn that $U' \cup W, U' \cup W'$ each contains a member of $\mathcal{C}$, so by Remark 2.3, $U' \cup W, U' \cup W'$ *are* members of $\mathcal{C}$. Repeating the argument on $\mathcal{C}/U \setminus U'$ instead of $\mathcal{C}'$ tells us that $U \cup W, U \cup W'$ are also members of $\mathcal{C}$. □

Let $\{I, J\}$ be an arbitrary reduction of $\mathcal{C}$. Pick two connected components of G disjoint from $I \cup J$ with bipartitions $\{U, U'\}, \{W, W'\}$, if any. We say that these two connected components are *I-linked* if at least one of $I \cup U \cup W, I \cup U \cup W', I \cup U' \cup W, I \cup U' \cup W'$ is a cover of $\mathcal{C}$. Similarly, we say that the two connected components are *J-linked* if at least one of $J \cup U \cup W, J \cup U \cup W', J \cup U' \cup W, J \cup U' \cup W'$ is a cover of $\mathcal{C}$. We say that the two connected components are *linked* if they are I- or J-linked.

Lemma 5.5. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V, let $G := G(\mathcal{C})$, and let $\{I, J\}$ be a reduction. Suppose $\{U, U'\}, \{W, W'\}$ are the bipartitions of two I-linked connected components of G. Then $G(\mathcal{C} \setminus I/J)$ has an edge between $U \cup U'$ and $W \cup W'$.*

Proof. Suppose otherwise. Let $J' := V - (I \cup U \cup U' \cup W \cup W')$, let $\mathcal{C}' := \mathcal{C} \setminus I/J'$, and let $G' := G(\mathcal{C}')$. (Note that $\mathcal{C}'$ is not necessarily the same as $\mathcal{C} \setminus I/J$.) Observe that $\mathcal{C}'$ has ground set $U \cup U' \cup W \cup W'$. As $\{I, J\}$ is a reduction and thus satisfies (R2), we have $\tau(\mathcal{C} \setminus I/J) \geq 2$, so $\tau(\mathcal{C}') \geq 2$ because $J \subseteq J'$. In particular, $G[U \cup U' \cup W \cup W'] \subseteq G'$, so $\mathcal{C}'$ is a clean tangled clutter where G' is a bipartite graph with at most two connected components.

Claim 1. *At least one of $U \cup W, U \cup W', U' \cup W, U' \cup W'$ is a cover of $\mathcal{C}'$.*

Proof of Claim. By assumption, the two connected components $\{U, U'\}, \{W, W'\}$ of G are I-linked, that is, at least one of $I \cup U \cup W, I \cup U \cup W', I \cup U' \cup W, I \cup U' \cup W'$ is a cover of $\mathcal{C}$. This immediately implies the claim. $\diamondsuit$

Claim 2. *G' has no edge between $U \cup U'$ and $W \cup W'$.*

Proof of Claim. Observe that the cardinality-two minimal covers of $\mathcal{C}'$ are precisely the same as the cardinality-two minimal covers of $\mathcal{C} \setminus I/J$ contained in $U \cup U' \cup W \cup W'$. As a result, $G' = G(\mathcal{C} \setminus I/J)[U \cup U' \cup W \cup W']$. Thus, our contrary assumption tells us that G' has no edge between $U \cup U'$ and $W \cup W'$, as claimed. $\diamondsuit$

In particular, G' has exactly two connected components, with bipartitions $\{U, U'\}, \{W, W'\}$. It therefore follows from Lemma 5.4 that $U \cup W, U \cup W', U' \cup W, U' \cup W'$ are all members of $\mathcal{C}'$. However, one of these members is necessarily disjoint from the cover guaranteed by Claim 1, a contradiction. $\square$

Lemma 5.6. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V, and let $G := G(\mathcal{C})$. Then the following statement holds:*

(a) Suppose $\{I, J\}$ is a reduction, and $\{U, U'\}$ is the bipartition of a connected component of G disjoint from $I \cup J$. Then none of $I \cup U, I \cup U', J \cup U, J \cup U'$ is a cover of $\mathcal{C}$.

(b) Suppose $\{I, J\}$ is an incrementally maximal reduction. Then every connected component of G disjoint from $I \cup J$ is linked to another connected component disjoint from $I \cup J$.

(c) An incrementally maximal reduction cannot be disjoint from exactly one connected component of G.

Proof. **(a)** Suppose otherwise. We may assume that $I \cup U$ is a cover of $\mathcal{C}$. Let $J' := V - (I \cup U \cup U')$ and $\mathcal{C}' := \mathcal{C} \setminus I/J'$. Observe that $\mathcal{C}'$ has ground set $U \cup U'$. As $\{I, J\}$ is a reduction and thus satisfies (R2), we have $\tau(\mathcal{C} \setminus I/J) = 2$, so $\tau(\mathcal{C}') \geq 2$ because $J \subseteq J'$. In particular, $G[U \cup U'] \subseteq G(\mathcal{C}')$, so $\mathcal{C}'$ is a clean tangled clutter where $G(\mathcal{C}')$ is a connected, bipartite graph whose bipartition is inevitably $\{U, U'\}$. However, as $I \cup U$ is a cover of $\mathcal{C}$, U must be a cover of $\mathcal{C}'$, a contradiction to Theorem 2.2.

(b) Let $\{U, U'\}$ be the bipartition of a connected component of G disjoint from $I \cup J$. Since $\{I, J\}$ is an incrementally maximal reduction, the set $\{I \cup U, J \cup U'\}$ is not a reduction. As (R1) is clearly satisfied, $\{I \cup U, J \cup U'\}$ must fail (R2). We may assume that $\tau(\mathcal{C} \setminus (I \cup U)/(J \cup U')) \leq 1$. By (a), $I \cup U$ is not a cover of $\mathcal{C}$, so $\tau(\mathcal{C} \setminus (I \cup U)/(J \cup U')) = 1$. That is, $\mathcal{C}$ has a minimal cover B such that $B \cap (J \cup U') = \emptyset$ and

$B - (I \cup U) = \{w\}$ for some $w \in V$. In particular, G has a connected component disjoint from $I \cup J \cup U \cup U'$ which contains w, say with bipartition $\{W, W'\}$ such that $w \in W$. Then $I \cup U \cup W \supseteq B$ is a cover of $\mathcal{C}$, so the two connected components $G[U \cup U'], G[W \cup W']$ are I-linked, as claimed.

(c) follows immediately from (b). □

We are now ready to prove the promised inequality:

Theorem 5.7. *Let $\mathcal{C}$ be a clean tangled clutter. Suppose $\{I, J\}$ is an incrementally maximal reduction that is proper. Then*

$$\operatorname{rank}(\mathcal{C} \setminus I/J) + \operatorname{rank}(\mathcal{C}/I \setminus J) \leq \frac{3}{2}(\operatorname{rank}(\mathcal{C}) - 1).$$

Proof. Denote by V the ground set of $\mathcal{C}$, and let $G := G(\mathcal{C})$ and $r := \operatorname{rank}(\mathcal{C})$. Let r' be the number of connected components of G disjoint from $I \cup J$. As $\{I, J\}$ is a proper reduction, $r' \geq 1$. In fact, as $\{I, J\}$ is incrementally maximal, $r' \geq 2$ by Lemma 5.6 (c). Also, it is clear that $r' \leq r - 1$.

Let $V'_1, \ldots, V'_{r'}$ be the vertex sets of the connected components of G disjoint from $I \cup J$, and let $V' := \bigcup_{i \in [r']} V'_i$. Let $G_1 := G(\mathcal{C} \setminus I/J)$ and $G_2 := G(\mathcal{C}/I \setminus J)$. Notice that G_1, G_2 both have vertex set V', and $G[V'] \subseteq G_1, G[V'] \subseteq G_2$. As a result, $G_1[V'_i]$ and $G_2[V'_i]$ are connected for each $i \in [r']$, implying in turn that $\operatorname{rank}(\mathcal{C} \setminus I/J) + \operatorname{rank}(\mathcal{C}/I \setminus J) \leq 2r'$. The upper bound here can be improved drastically. For each V'_i, there exists a different V'_j linked to it by Lemma 5.6 (b), so there exists an edge between V'_i, V'_j in G_1 or G_2 by Lemma 5.5. Subsequently, $\operatorname{rank}(\mathcal{C} \setminus I/J) + \operatorname{rank}(\mathcal{C}/I \setminus J) \leq \frac{3}{2}r'$. Since $r' \leq r - 1$, Theorem 5.7 follows. □

5.4 Algorithm 2 and the proof of Theorem 1.7

Let us modify Algorithm 1 as follows:

Algorithm 2: A second algorithm for finding a dyadic fractional packing of value two

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Input: A filter oracle for a clean tangled clutter $\mathcal{C}$

Output: A dyadic fractional packing y of value two, provided explicitly

Initialize: Let $\{I, J\}$ be an incrementally maximal reduction;

if $\{I, J\}$ *is a non-proper reduction* **then**

  Let $y \in \mathbb{R}^{\mathcal{C}}_{+}$ be the incidence vector of the packing I, J (see (R2b));

  Output y.

else

  Let $\mathcal{C}_1 := \mathcal{C} \setminus I/J$ and $\mathcal{C}_2 := \mathcal{C}/I \setminus J$;

  Run Algorithm 2 on $\mathcal{C}_1$ (the filter oracle for $\mathcal{C}$ gives one for $\mathcal{C}_1$ by Remark 4.2) with output z;

  Run Algorithm 2 on $\mathcal{C}_2$ (the filter oracle for $\mathcal{C}$ gives one for $\mathcal{C}_2$ by Remark 4.2) with output z';

  Define y as in Lemma 5.2;

  Output y.

Lemma 5.8. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V inputted via a filter oracle. Then the following statements hold:*

(a) One can test whether or not a given $\{I, J\}$ is a reduction with $O(|V|^2)$ calls to the oracle and with total running time $O(|V|^2)$.

(b) An incrementally maximal reduction can be found with $O(|V|^3)$ calls to the oracle and with running time $O(|V|^2)$ per call.

Proof. Let $G := G(\mathcal{C})$, $r := \mathrm{rank}(\mathcal{C})$, and $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G; these can be found with at most $|V|^2$ calls to the oracle by Remark 4.1, and with running time $O(|V|^2)$. **(a)** Thus (R1) can be tested in time $O(|V|^2)$. For (R2), notice first that the filter oracle for $\mathcal{C}$ yields a filter oracle for $b(\mathcal{C})$ by Remark 4.3. Notice further that (R2) is equivalent to the following: Neither I nor J is a cover of $\mathcal{C}$, and neither $I \cup \{v\}$ nor $J \cup \{v\}$ is a cover of $\mathcal{C}$ for any $v \in V - (I \cup J)$. As a result, (R2) can be tested by making $2 + 2 \cdot |V - (I \cup J)|$ queries to the filter oracle for $b(\mathcal{C})$. Hence, testing whether or not $\{I, J\}$ is a reduction can be carried out with $O(|V|^2)$ calls to the oracle, and with running time $O(|V|^2)$. **(b)** We leave this as an exercise for the reader. □

We are now ready to analyze Algorithm 2, in turn proving Theorem 1.7:

Proof of Theorem 1.7. Denote by T the enumeration tree of Algorithm 2. Then T is a rooted binary tree where each node is labeled with the rank of the corresponding clutter. Observe that the root has rank r, and the rank of each child is at least one less than the rank of the parent by (R4). Moreover, by Theorem 5.7, if an inner node has rank x, then the sum of the ranks of its two children is at most $\frac{3}{2}(x-1)$. It therefore follows from Theorem 4.5 (b) that T has height $\leq r-1$, its number of leaves is $r^{O(\log r)}$, and the total number of nodes is also $r^{O(\log r)}$.

As the tree has height $\leq r-1$ and the number of leaves of the tree is at most $r^{O(\log r)}$, it follows from Lemma 5.2 that $y^\star$ is $\frac{1}{2^{r-1}}$-integral and its support has cardinality $r^{O(\log r)}$.

The initialization step takes $O(|V|^3)$ calls to the oracle with running time $O(|V|^2)$ per call by Lemma 5.8 (b). Thus, for each node of the enumeration tree T, $O(|V|^3)$ calls to the oracle are made. Since the number of nodes of the tree is $r^{O(\log r)}$, the algorithm makes $r^{O(\log r)} \cdot O(|V|^3)$ calls to the oracle and has running time $O(|V|^2)$ per call. □

6 Algorithm 2 on binary clutters

In this section, we analyze the performance of Algorithm 2 on the class of binary clutters, showing that the algorithm has polynomial guarantees for this class. By and large, the improved guarantee is due to a balanced rank drop in the enumeration tree in the binary case (see Theorem 6.2 below) as opposed to a possibly skewed rank drop in the enumeration tree in the general case (see Theorem 5.7 above).

Let $\mathcal{C}$ be a clutter, and let u, v be distinct elements. We say that u, v are *duplicates in* $\mathcal{C}$ if every member containing u also contains v, and vice versa. We say that u, v are *replicates in* $\mathcal{C}$ if every member contains at most one of u, v, and whenever C is a member containing exactly one of u, v, then $C \triangle \{u, v\}$ is also a member. It can be readily checked that a pair of duplicates in a clutter correspond to a pair of replicates in the blocker, and vice versa. We need the following lemma:

Lemma 6.1. *Let $\mathcal{C}$ be a binary tangled clutter, and let $G := G(\mathcal{C})$. Then the following statements hold:*

(a) Every minimum cover of $\mathcal{C}$ is a transversal.

(b) Every connected component of G is a complete bipartite graph.

(c) Every non-minimum minimal cover of $\mathcal{C}$ contains at most one vertex from each connected component of G.

(d) Let B be a non-minimum minimal cover of $\mathcal{C}$. For each $u \in B$, let $f(u)$ be a neighbour of u in G. Then for every subset $T \subseteq B$ of even cardinality, $B \triangle \{u, f(u) : u \in T\}$ is also a cover of $\mathcal{C}$.

(e) Let $\{I, J\}$ be a reduction. Then two connected components of G are I-linked if and only if they are J-linked.

Proof. **(a)** Let $\{u, v\}$ be a minimum cover of $\mathcal{C}$. As $\mathcal{C}$ is binary, $|\{u, v\} \cap C|$ must be odd and therefore 1 for any $C \in \mathcal{C}$, implying in turn that $\{u, v\}$ is a transversal of $\mathcal{C}$.

(b) If $\{u, v\}, \{v, w\}$ are minimum covers of $\mathcal{C}$, then u, w must be duplicates in $\mathcal{C}$ by (a). Subsequently, if $\{u, v\}, \{v, w\}, \{w, t\}$ are minimum covers of $\mathcal{C}$, then so is $\{u, t\}$. This observation proves (b).

(c) Let $\{U, U'\}$ be the bipartition of a connected component of G. We just saw that the elements in U (resp. U') are duplicates in $\mathcal{C}$, and therefore replicates in $b(\mathcal{C})$. Thus, every minimal cover of $\mathcal{C}$ contains at most one vertex from each of U, U'. It now follows from (b) that every non-minimum minimal cover contains at most one vertex from $U \cup U'$.

(d) By definition, $\{u, f(u)\}$ is a minimal cover for each $u \in B$. As $\mathcal{C}$ is a binary clutter, so is its blocker, implying that the symmetric difference of any odd number of minimal covers is also a cover. As a result, the set $B \triangle \{u, f(u) : u \in T\} = B \triangle (\triangle_{u \in T} \{u, f(u)\})$ is a cover.

(e) It suffices to prove $(\Rightarrow)$. Suppose $\{U, U'\}, \{W, W'\}$ are two connected components of G disjoint from $I \cup J$ that are I-linked. That is, one of $I \cup U \cup W, I \cup U \cup W', I \cup U' \cup W, I \cup U' \cup W'$, say $I \cup U \cup W$, is a cover of $\mathcal{C}$. Let B be a minimal cover contained in $I \cup U \cup W$. Since $\{I, J\}$ is a reduction, neither $I \cup U$ nor $I \cup W$ is a cover by Lemma 5.6 (a), so $B \cap U \neq \emptyset$ and $B \cap W \neq \emptyset$. As B is a stable set in G, it is a non-minimum cover of $\mathcal{C}$, so by (c), B picks at most one vertex from every connected component. For each $u \in B$, let $f(u)$ be a neighbour of u in G. Pick a set T such that $I \cap B \subseteq T \subseteq B$ and $|T|$ is even – this is possible since $|I \cap B| < |B|$. Then $B \triangle \{u, f(u) : u \in T\}$ is a cover of $\mathcal{C}$ by (d). As one of $J \cup U \cup W, J \cup U \cup W', J \cup U' \cup W, J \cup U' \cup W'$ is a superset of $B \triangle \{u, f(u) : u \in T\}$ and is therefore a cover, it follows that $\{U, U'\}, \{W, W'\}$ are J-linked. □

Lemma 6.1 (e) enables us to prove the following strengthening of Theorem 5.7.

Theorem 6.2. *Let $\mathcal{C}$ be a binary tangled clutter. Suppose $\{I, J\}$ is an incrementally maximal reduction that is proper. Then*

$$\operatorname{rank}(\mathcal{C} \setminus I/J) \leq \frac{1}{2}(\operatorname{rank}(\mathcal{C}) - 1) \quad \textit{and} \quad \operatorname{rank}(\mathcal{C}/I \setminus J) \leq \frac{1}{2}(\operatorname{rank}(\mathcal{C}) - 1).$$

Proof. We only prove the first inequality as the second inequality's proof follows a similar argument. By Lemma 5.6 (b) and Lemma 6.1 (e),

> ($\star$) every connected component disjoint from $I \cup J$ is I-linked to another connected component disjoint from $I \cup J$.

Let $V_1', \ldots, V_{r'}'$ be the vertex sets of the connected components of $G := G(\mathcal{C})$ disjoint from $I \cup J$, and let $V' := \bigcup_{i \in [r']} V_i'$. Let $G' := G(\mathcal{C} \setminus I/J)$. Then G' has vertex set V', and $G[V'] \subseteq G'$. As a result, $G'[V_i']$ is connected for each $i \in [r']$, implying in turn that

$$\operatorname{rank}(\mathcal{C} \setminus I/J) \leq r'.$$

The upper bound above can be improved drastically. For each V_i', there exists a different V_j' that is I-linked to it by ($\star$), so there exists an edge between V_i', V_j' in G' by Lemma 5.5. Subsequently,

$$\operatorname{rank}(\mathcal{C} \setminus I/J) \leq \frac{1}{2} r'.$$

Since $r' \leq \operatorname{rank}(\mathcal{C}) - 1$, Theorem 6.2 follows. □

We are now ready to prove that Algorithm 2 has a polynomial running time performance guarantee on binary tangled clutters:

Theorem 6.3. *Let $\mathcal{C}$ be a binary tangled clutter of rank r over ground set V inputted via a filter oracle. Then Algorithm 2 outputs a fractional packing $y^\star$ of value two that is $\frac{1}{2^{k-1}}$-integral, for some integer k such that $1 \leq k \leq \log(r+1)$, and whose support has cardinality at most $r+1$. Moreover, the algorithm makes $r \cdot O(|V|^3)$ calls to the oracle and has running time $O(|V|^2)$ per call.*

Proof. Denote by T the enumeration tree of Algorithm 2. Then T is a rooted binary tree where each node is labeled with the rank of the corresponding clutter. Observe that the root has rank r, and the rank of each child is at least one less than the rank of the parent by (R4). Moreover, by Theorem 6.2, if an inner node has rank x, then each of its children has rank at most $\frac{1}{2}(x-1)$. It therefore follows from Theorem 4.5 (c) that T has height $\leq \log{(r+1)} - 1$, its number of leaves is $\leq \frac{r+1}{2}$, and the total number of nodes is $\leq r$.

As the tree has height $\leq \log{(r+1)} - 1$ and the number of leaves of the tree is at most $\leq \frac{r+1}{2}$, it follows from Lemma 5.2 that $y^\star$ is $\frac{1}{2^{k-1}}$-integral, for some integer k such that $1 \leq k \leq \log(r+1)$, and its support has cardinality $\leq 2 \times \frac{r+1}{2} = r + 1$.

The initialization step takes $O(|V|^3)$ calls to the oracle with running time $O(|V|^2)$ per call by Lemma 5.8 (b). Thus, for each node of the enumeration tree T, $O(|V|^3)$ calls to the oracle are made. Since the number of nodes of the tree is $\leq r$, the algorithm makes $r \cdot O(|V|^3)$ calls to the oracle and has running time $O(|V|^2)$ per call. □

In closing, we note that the guarantee on the fractionality and the support cardinality from Theorem 6.3 is best possible. This is because of Theorem 3.4, combined with the fact that the cuboid of $\mathrm{cocycle}(PG(k-1,2))$ is a binary tangled clutter of rank $2^k - 1$.

(As binary tangled clutters arise from binary matroids, one may wonder what Algorithm 2 accomplishes explicitly on binary matroids – we address this in the appendix. Along the way, we see concrete examples of reductions and incrementally maximal reductions.)

7 Concluding remarks

Let us conclude with some discussion about improving Theorem 1.4 and Theorem 1.7, and also extensions and restrictions of Conjecture 1.1.

7.1 Two conjectures on dyadic fractional packings of value two

Let $\mathcal{C}$ be a clean tangled clutter over ground set V inputted via a filter oracle. Algorithm 2 successfully outputs a dyadic fractional packing of $\mathcal{C}$ of value two. However, the number of calls made to the oracle is quasi-polynomial in r. The bottleneck in achieving a polynomial dependence on r is controlling the height of the enumeration tree or its proxy, the fractionality of the dyadic fractional packing. We conjecture the following:

Conjecture 7.1. *Let $\mathcal{C}$ be a clean tangled clutter. Then for some integer $k \geq 1$ such that $2^k - 1 \leq \mathrm{rank}(\mathcal{C})$, there exists a $\frac{1}{2^{k-1}}$-integral packing of value two.*

This conjecture, if true, would imply Conjecture 3.5 from §3. Conjecture 7.1 holds for binary tangled clutters by Theorem 6.3. As further evidence for the conjecture, the following special case is proved in a sequel paper:

Theorem 7.2 ([8])**.** *Let $\mathcal{C}$ be a clean tangled clutter with a unique fractional packing y of value two. Then for some integer $k \geq 1$, y is $\frac{1}{2^{k-1}}$-integral and $\mathrm{rank}(\mathcal{C}) = 2^k - 1$.*

Conjecture 7.1, along with the evidence for it, leads us to believe the following:

Conjecture 7.3. *There exists an algorithm that, given a clean tangled clutter over ground set V via a filter oracle, outputs a dyadic fractional packing of value two and its number of calls to the oracle is upper bounded by a polynomial in $|V|$.*

Observe that this conjecture holds for binary tangled clutters by Theorem 6.3, and this is achieved by Algorithm 2. In fact, as far as we know, Algorithm 2 is a worthy candidate for proving the conjecture in the general case of clean tangled clutters.

We note that Conjecture 7.1 and Conjecture 7.3 are open for ideal tangled clutters.

Using recent results on *dyadic linear programming*, the authors prove the following variant of Conjecture 7.3:

Theorem 7.4 ([1])**.** *There exists an algorithm that, given a clean tangled clutter $\mathcal{C}$ over ground set V, outputs a dyadic fractional packing of value two and its running time is upper bounded by a polynomial in $|V|$ and $|\mathcal{C}|$.*

7.2 Possible extensions and restrictions of Conjecture 1.1

Let us look back at the dual programs below for a clutter $\mathcal{C}$ over ground set V:

$$(P)\quad \begin{array}{ll} \min & \mathbf{1}^\top x \\ \text{s.t.} & x(C) \geq 1 \; \forall C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array} \qquad (D)\quad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum(y_C : v \in C, C \in \mathcal{C}) \leq 1 \; \forall v \in V \\ & y \geq \mathbf{0}. \end{array}$$

Recall that Conjecture 1.1 predicts that for ideal clutters $\mathcal{C}$, (D) must have an optimal solution that is dyadic. Given Theorem 1.5, a natural wondering is whether Conjecture 1.1 can be extended to all clean clutters $\mathcal{C}$. The answer, however, turns out to be no. It does not even hold for the other two basic classes of clean clutters – binary clutters and clutters without an intersecting minor – the reason being that the joint optimal value of (P) and (D) can be a non-dyadic rational number. For instance, consider the two clutters

$$\mathbb{L}_7 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,5,6\},\{2,4,7\},\{3,4,6\},\{3,5,7\}\}$$
$$\mathcal{C}_8^3 = \{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5,6\},\{5,6,7\},\{6,7,8\},\{7,8,1\},\{8,1,2\}\}.$$

Both of these clutters are *minimally non-ideal*, that is, each clutter is non-ideal but every proper minor of it is ideal. It can be readily checked that for the first clutter, the joint optimal value of (P) and (D) is the non-dyadic number $\frac{7}{3}$, while for the second clutter this value is the non-dyadic number $\frac{8}{3}$. As $\mathbb{L}_7$ is also a binary clutter, it follows that the analogue of Conjecture 1.1 does not hold for binary clutters. The reader can also verify that $\mathcal{C}_8^3$ has no intersecting minor, so the analogue of Conjecture 1.1 does not hold for clutters without an intersecting minor.

What about restricting Conjecture 1.1 to special cases? A special case is obtained by focusing on ideal clutters without an intersecting minor:

Conjecture 7.5. *Every ideal clutter without an intersecting minor has an optimal fractional packing that is dyadic.*

In fact, the $\tau = 2$ Conjecture [14], mentioned in the introduction, would imply that every ideal clutter without an intersecting minor has an optimal fractional packing that is integral. Given this observation, Conjecture 7.5 is quite intriguing as it seems to open a backdoor to approaching the $\tau = 2$ Conjecture.

Another quite intriguing special case of Conjecture 1.1 is obtained by focusing on ideal binary clutters:

Conjecture 7.6. *Every ideal binary clutter has an optimal fractional packing that is dyadic.*

Other than the clutter of T-cuts of a graph, this conjecture is known to hold for three other classes. Very recently, the conjecture was proved for the clutter of T-joins of a graph [7]. Given a signed graph, if the (binary) clutter of *odd circuits* is ideal, then the clutter must have an optimal fractional packing that is $\frac{1}{2}$-integral [21]. More generally, given a signed graph and (possibly equal) vertices s,t, if the (binary) clutter of *odd st-walks* is ideal, then the clutter must have an optimal fractional packing that is $\frac{1}{2}$-integral [9, 10].

We have considered ideal clutters without an intersecting minor, as well as ideal binary clutters. The reader might wonder about binary clutters without an intersecting minor. Recall Q_6 from §1, the clutter whose elements are the edges and whose members are the triangles of K_4. Then Q_6 is a binary clutter that is intersecting. Seymour proved that if $\mathcal{C}$ is a binary clutter without a Q_6 minor, then the associated set covering linear system must be totally dual integral [32]. In particular, every binary clutter without an intersecting minor has an optimal fractional packing that is integral, thereby verifying Conjecture 7.6 for this special class.

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A Affine flats and cocycle covers in binary matroids

Every binary tangled clutter is a duplication of the cuboid of the cocycle space of a loopless binary matroid [5]. In this section, we give a quick and casual overview of what Algorithm 2 accomplishes directly on the binary matroid itself. We follow Oxley's terminology of binary matroids [29].

Let M be a loopless binary matroid over ground set E. Let $\mathcal{C}$ be the clutter over ground set $\{e : e \in E\} \cup \{\tilde{e} : e \in E\}$ that has a member of the form $D \cup \{\tilde{e} : e \in E - D\}$ per cocycle D of M. Observe that $\mathcal{C} = \text{cuboid}(\text{cocycle}(M))$, where $\text{cocycle}(M) = \{\chi_D : D \text{ is a cocycle of } M\}$ denotes the cocycle space of M. Since M is a loopless binary matroid, $\text{cocycle}(M)$ is a binary space whose points do not agree on a coordinate, so $\mathcal{C}$ is a binary tangled clutter.

A fractional packing of value two in $\mathcal{C}$ corresponds entry-wise to a *fractional* 2*-cocycle* 1*-cover* of M, i.e. an assignment $y_D \in \mathbb{R}_+$ to every cocycle D of M such that $\mathbf{1}^\top y = 2$ and $\sum (y_D : e \in D) = 1$ for every element $e \in E$.

Algorithm 2 looks for a dyadic fractional packing of value two in $\mathcal{C}$ via reductions. What do these correspond to in M? To answer this question, we need the notion of affine flats in binary matroids.

Let $A \subseteq E$. We say that A is *affine in* M if every cycle contained in A has even cardinality, and that A is a *flat of* M if there exists no cycle C such that $|C - A| = 1$. An *affine flat of* M is a flat that is affine. An affine flat of M is *incrementally maximal* if it is not contained in an affine flat that has just one new parallel class.

If M is a projective geometry, for instance, then every affine flat would have cardinality at most one, because every pair of elements appear in a circuit of cardinality three. As another example, if M is the cycle matroid of K_5, whose edges are labeled as in Figure 3, then any nonempty matching of K_5 forms an affine flat of M, while any maximal matching forms an incrementally maximal affine flat. On the other hand, if M is the dual of the cycle matroid of K_5, then any circuit of K_5 forms an affine flat of M; it is affine because no cut of K_5 is contained in a circuit, and it is a flat because deleting the edges of a circuit of K_5 does not create a bridge. The circuit $\{1, 2, 9\}$, for example, is not incrementally maximal because it is contained in the affine flat $\{1, 2, 9, 4\}$. The circuit $\{1, 2, 3, 4, 5\}$, on the other hand, is incrementally maximal. This example is particularly interesting because the entire edge set $\{1, \ldots, 10\}$ is another incrementally maximal affine flat of M, thereby showing that one incrementally maximal affine flat may be contained in another one.

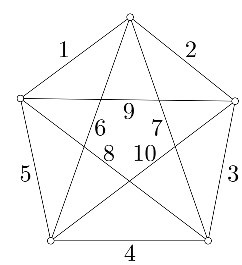

Figure 3: The complete graph K_5

We also need two functions. The first one is an involution of the form $f : E \cup \widetilde{E} \to E \cup \widetilde{E}$ where $f(e) := \tilde{e}$ and $f(\tilde{e}) := e$ for all $e \in E$. The second one is a projection $p : E \cup \widetilde{E} \to E$ defined as $p(e) := e$ and $p(\tilde{e}) := e$ for all $e \in E$. We follow the convention that for a subset X, $f(X) := \{f(x) : x \in X\}$ and $p(X) := \{p(x) : x \in X\}$.

We prove the following theorem in §A.1, linking (incrementally maximal) affine flats in binary matroids to (incrementally maximal) reductions in binary tangled clutters:

Theorem A.1. *Let M be a loopless binary matroid, and let $\mathcal{C} := \text{cuboid}(\text{cocycle}(M))$. Then the following statements hold:*

(a) if A is an affine flat in M, then $\{A, f(A)\}$ is a reduction in $\mathcal{C}$,

(b) if $\{I, f(I)\}$ is a reduction in $\mathcal{C}$, then $p(I)$ is an affine flat in M,

(c) if A is an incrementally maximal affine flat in M, then $\{A, f(A)\}$ is an incrementally maximal reduction in $\mathcal{C}$.

Consider a fine-tuned version of Algorithm 2 where at every iteration, the incrementally maximal reduction comes from an incrementally maximal affine flat of the binary matroid, per Theorem A.1 (c). We may then reinterpret the algorithm as one that looks for a *cocycle cover of* M, i.e. a family $\{D_1, \ldots, D_k\}$ of cocycles of M whose union is E; see Algorithm 3. Given the cocycle cover, a dyadic fractional cocycle cover can be obtained as follows: First, consider the subspace $S := \langle \chi_{D_i} : i = 1, \ldots, k \rangle$ over $GF(2)$ generated by the cocycles $D_1, \ldots, D_k$. Then, given that S has $GF(2)$-rank $r \leq k$ and therefore $|S| = 2^r$, a dyadic fractional cocycle cover is obtained by assigning $\frac{1}{2^{r-1}}$ to every cocycle of M whose incidence vector appears in S.

Algorithm 3: An algorithm for finding a cocycle cover in a binary matroid, derived from Algorithm 2

Input: A loopless binary matroid M over ground set E

Output: A cocycle cover $\mathcal{D}$ of M

Initialize: Let A be an incrementally maximal affine flat;

if $A = E$ **then**
  Output $\mathcal{D} := \{A\}$.
else
  Let D be a cocycle of M containing A (see Remark A.2 (b));
  Run Algorithm 3 on M/A with output $\mathcal{D}$;
  Output $\mathcal{D} \cup \{D\}$.

A.1 Reductions from affine flats

In this subsection, we prove Theorem A.1. We need the following remark:

Remark A.2 (see [29], §9)**.** *Let M be a binary matroid over ground set E, and let $A \subseteq E$. Then the following statements hold:*

(a) If every cycle has even cardinality, then E is a cocycle.

(b) If A is affine, then it is contained in a cocycle.

(c) If $e \in E$ is not contained in any cycle, then $\{e\}$ is a cocycle.

(d) If $e \in A$ and every cycle in A avoids e, then there exists a cocycle D such that $D \cap A = \{e\}$.

Proof. **(a)** As every cycle has even cardinality, $|E \cap C|$ is even for every cycle C, implying in turn that E is a cocycle. **(b)** As A is affine, every cycle of $M \setminus (E - A)$ has even cardinality, so by (a), A forms a cocycle of $M \setminus (E - A)$, implying in turn that A is contained in a cocycle of M. **(c)** As $|\{e\} \cap C|$ is even for every cycle C, it follows that $\{e\}$ is a cocycle. **(d)** Every cycle of $M \setminus (E - A)$ avoids e, so by (c), $\{e\}$ is a cocycle of $M \setminus (E - A)$, implying in turn that there exists a cocycle D of M such that $D \cap A = \{e\}$. □

Let M be a loopless binary matroid over ground set E. Let $\mathcal{C}$ be the clutter over ground set $\{e : e \in E\} \cup \{\tilde{e} : e \in E\}$ that has a member of the form $D \cup \{\tilde{e} : e \in E - D\}$ per cocycle D of M. Recall that $\mathcal{C} = \mathrm{cuboid}(\mathrm{cocycle}(M))$, $\mathcal{C}$ is a binary tangled clutter, $f : E \cup \widetilde{E} \to E \cup \widetilde{E}$ is an involution where $f(e) = \tilde{e}$ and $f(\tilde{e}) = e$ for all $e \in E$, and $p : E \cup \widetilde{E} \to E$ is a projection where $p(e) = e$ and $p(\tilde{e}) = e$ for all $e \in E$.

Remark A.3. *Let M be a loopless binary matroid over ground set E, and let $\mathcal{C} := \mathrm{cuboid}(\mathrm{cocycle}(M))$. Then $e, e' \in E \cup \widetilde{E}$ are in the same connected component of $G(\mathcal{C})$ if, and only if, $p(e), p(e')$ are either equal or parallel in M.*

Proof. ($\Rightarrow$) Assume in the first case that $e \in E$ and $e' \in \widetilde{E}$. It then follows from Lemma 6.1 (b) that $\{e, e'\}$ is a minimal cover of $\mathcal{C}$, so by Lemma 6.1 (a), $\{e, e'\}$ is a transversal, implying in turn that $f(e), e'$ are either equal or duplicates in $\mathcal{C}$, implying in turn that $p(e), p(e')$ are either equal or parallel in M, as required. The remaining cases are treated similarly, so we leave their verification to the reader. ($\Leftarrow$) is also left to the reader. □

We are now ready to prove Theorem A.1:

Proof of Theorem A.1. **(a)** Let A be an affine flat in M. As a flat, if A contains an element, then it must contain all the elements parallel to it. Thus, it follows from Remark A.3 that $\{A, f(A)\}$ satisfies (R1). It remains to show that (R2) is also satisfied.

Claim 1. *Neither A nor $f(A)$ is a cover of $\mathcal{C}$. That is, $\tau(\mathcal{C} \setminus A / f(A)) \geq 1$ and $\tau(\mathcal{C} \setminus f(A)/A) \geq 1$.*

Proof of Claim. Since A is disjoint from $\widetilde{E} \in \mathcal{C}$, it is not a cover. As an affine set, A is contained in a cocycle D of M, by Remark A.2 (b). As a result, $f(A)$ is disjoint from $D \cup f(E - D) \in \mathcal{C}$, implying in turn that $f(A)$ is not a cover. ♢

Claim 2. *For $c \in E - A$, neither $A \cup \{c\}$ nor $A \cup \{\tilde{c}\}$ is a cover of $\mathcal{C}$.*

Proof of Claim. Since $A \cup \{c\}$ is disjoint from $\widetilde{E} \in \mathcal{C}$, it is not a cover. Since A is a flat, every cycle of M contained in $A \cup \{c\}$ avoids c, implying by Remark A.2 (d) that there exists a cocycle D such that $D \cap (A \cup \{c\}) = \{c\}$. Consequently, $A \cup \{\tilde{c}\}$ is disjoint from $D \cup f(E - D) \in \mathcal{C}$, so $A \cup \{\tilde{c}\}$ is not a cover of $\mathcal{C}$. ♢

Claim 3. *For $c \in E - A$, neither $f(A) \cup \{c\}$ nor $f(A) \cup \{\tilde{c}\}$ is a cover of $\mathcal{C}$.*

Proof of Claim. Suppose for a contradiction that $f(A) \cup \{e\}$ is a cover of $\mathcal{C}$ for some $e \in \{c, \tilde{c}\}$. Let B be a minimal cover of $\mathcal{C}$ contained in $f(A) \cup \{e\}$. By Claim 1, $e \in B$. Let T be an even cardinality set such that $B - \{e\} \subseteq T \subseteq B$. It then follows from Lemma 6.1 (e) that $B \triangle \{u, f(u) : u \in T\}$ is also a cover of $\mathcal{C}$. However, $B \triangle \{u, f(u) : u \in T\}$ is a subset of $A \cup \{f(e)\}$, so $A \cup \{f(e)\}$ which is either $A \cup \{c\}$ or $A \cup \{\tilde{c}\}$, is also a cover of $\mathcal{C}$, a contradiction to Claim 2. ◊

Claims 2 and 3 imply that $\tau(\mathcal{C} \setminus A / f(A)) \geq 2$ and $\tau(\mathcal{C}/A \setminus f(A)) \geq 2$, so $\{A, f(A)\}$ satisfies (R2), as required.

(b) Suppose $\{I, f(I)\}$ is a reduction in $\mathcal{C}$.

Claim 4. *Let C be a cycle of M, and K be a subset of $C \cup \widetilde{C}$ that picks exactly one of $e, \tilde{e}$ for every $e \in C$. Then*

$$|K \cap (D \cup f(E - D))| \equiv |K \cap \widetilde{C}| \pmod 2 \quad \forall \text{ cocycles } D \text{ of } M.$$

Proof of Claim. Let D be a cocycle of M. Then

$$\begin{aligned} K \cap (D \cup f(E - D)) &= (K \cap D) \cup (K \cap f(E - D)) \\ &= (K \cap D \cap C) \cup (K \cap f(E - D) \cap f(C)) \\ &\qquad \text{because } K \cap D \subseteq C \text{ and } K \cap f(E - D) \subseteq f(C) \\ &= K \cap [(C \cap D) \cup f(C - D)]. \end{aligned}$$

As C is a cycle and D a cocycle, $|C \cap D|$ is even, implying that $|K \cap (C \cap D)|, |K \cap f(C \cap D)|$ have the same parity, and so

$$|K \cap [(C \cap D) \cup f(C - D)]| \equiv |K \cap [f(C \cap D) \cup f(C - D)]| \equiv |K \cap f(C)| \pmod 2.$$

Thus,

$$|K \cap (D \cup f(E - D))| \equiv |K \cap f(C)| \pmod 2$$

as claimed. ◊

Claim 5. *$p(I)$ is affine in M.*

Proof of Claim. Suppose otherwise. Then $p(I)$ contains a cycle C of M of odd cardinality. Let $I' := I \cap (C \cup f(C))$. Observe that $f(I') = f(I) \cap (C \cup f(C))$, $I' \cup f(I') = C \cup f(C)$, and for every $e \in C$, I' picks one of $e, \tilde{e}$ and $f(I')$ picks the other element. As a result, we may apply Claim 4 to $K = I', f(I')$ to conclude that for every cocycle D of M,

$$|I' \cap (D \cup f(E - D))| \equiv |I' \cap f(C)| \pmod 2 \quad \text{and} \quad |f(I') \cap (D \cup f(E - D))| \equiv |f(I') \cap f(C)| \pmod 2$$

As

$$|I' \cap f(C)| + |f(I') \cap f(C)| \equiv |f(C)| \equiv |C| \equiv 1 \pmod 2$$

we get that one of $I', f(I')$ intersects every member of $\mathcal{C}$ in an odd number of elements, implying that one of $I', f(I')$ is a cover of $\mathcal{C}$. This in turn implies that one of $I, f(I)$ is a cover of $\mathcal{C}$, so either $\tau(\mathcal{C} \setminus I/f(I)) = 0$ or $\tau(\mathcal{C}/I \setminus f(I)) = 0$, a contradiction as $\{I, f(I)\}$ is a reduction in $\mathcal{C}$. $\lozenge$

Claim 6. *$p(I)$ is a flat of M.*

Proof of Claim. Suppose otherwise. Then there exists a cycle C of M such that $C - p(I) = \{c\}$ for some $c \in E$. Note that $(I \cup f(I)) \cap \{c, \tilde{c}\} = \emptyset$. Let $I' := I \cap (C \cup f(C))$. Observe that for every $e \in C - \{c\}$, I' picks exactly one of $e, \tilde{e}$. We may therefore apply Claim 4 to $K = I' \cup \{c\}, I' \cup \{\tilde{c}\}$ to conclude that for every cocycle D of M,

$$|(I' \cup \{c\}) \cap (D \cup f(E - D))| \equiv |(I' \cup \{c\}) \cap f(C)| \equiv |I' \cap f(C)| \pmod 2$$
$$|(I' \cup \{\tilde{c}\}) \cap (D \cup f(E - D))| \equiv |(I' \cup \{\tilde{c}\}) \cap f(C)| \equiv |I' \cap f(C)| + 1 \pmod 2.$$

As a result, one of $I' \cup \{c\}, I' \cup \{\tilde{c}\}$ intersects every member of $\mathcal{C}$ in an odd number of elements and is therefore a cover of $\mathcal{C}$. Subsequently, one of $I \cup \{c\}, I \cup \{\tilde{c}\}$ is a cover of $\mathcal{C}$. Since $f(I) \cap \{c, \tilde{c}\} = \emptyset$, we have that $\tau(\mathcal{C} \setminus I/f(I)) \leq 1$, which is a contradiction as $\{I, f(I)\}$ is a reduction of $\mathcal{C}$. $\lozenge$

Claims 5 and 6 finish the proof of (a).

(c) Let A be an incrementally maximal affine flat in M. By (a), $\{A, f(A)\}$ is a reduction of $\mathcal{C}$. Suppose for a contradiction that $\{A, f(A)\}$ is not incrementally maximal, that is, $\{A \cup \{e\}, f(A \cup \{e\})\}$ is a reduction for some $e \in (E \cup \widetilde{E}) - (A \cup f(A))$. It follows from (b) that $p(A \cup \{e\})$, which is a superset of A of cardinality one larger, is an affine flat, a contradiction to the incremental maximality of A. $\square$