

Cosigning Crossing Families and Outer-Planar Gadgets

Ahmad Abdi¹[0000-0002-3008-4167], Mahsa Dalirrooyfard¹[0009-0004-8343-9840]^{*},
and Meike Neuwöhner²[0000-0002-3664-3687]^{**}

¹ LSE, Houghton Street, London, WC2A 2AE, England, United Kingdom
{a.abdi1,m.dalirrooy-fard}@lse.ac.uk

² CNRS & DIENS, ENS Paris - PSL Research University, 75005 Paris, France
meike.neuwohner@ens.fr

Abstract. Let F be a crossing family over ground set V , that is, for any two sets $U, W \in F$ with nonempty intersection and proper union, both sets $U \cap W, U \cup W$ are in F . Let $\sigma : V \rightarrow \{+, -\}$ be a signing. We call σ a *cosigning* if every set includes a positive element and excludes a negative element. It is $\cap\cup$ -closed if every pairwise nonempty intersection and co-intersection include positive and negative elements, respectively. We characterize the existence of ($\cap\cup$ -closed) cosignings σ through necessary and sufficient conditions. Our proofs are algorithmic and lead to elegant ‘forcing’ algorithms for finding σ , reminiscent of the Cameron-Edmonds algorithm for bicoloring balanced hypergraphs. We prove that the algorithms run in polynomial time, and further, the cosigning algorithm can be run in oracle polynomial time through an application of submodular function minimization.

Cosigned crossing families arise naturally in digraphs with vertex set V comprised of sources and sinks, where every set in F is *covered* by an incoming arc. Under mild and necessary conditions, we build an outer-planar arc covering of F when the vertices are placed around a circle. These gadgets are then used to find disjoint dijoins in 0, 1-weighted planar digraphs when the weight-1 arcs form a connected component that is not necessarily spanning.

Keywords: graph orientation · crossing family · submodular function minimization · disjoint dijoins · planar graph · polynomial algorithm.

1 Introduction

Let V be a finite set, and let $F \subseteq 2^V \setminus \{\emptyset, V\}$ be a *crossing family* of subsets of V , that is, $U \cap W, U \cup W \in F$ for any two sets $U, W \in F$ that *cross*, i.e., $U \cap W \neq \emptyset$ and $U \cup W \neq V$. Crossing families form a basic class of set families amenable to

^{*} Corresponding author

This paper is a part of the PhD Thesis of M. Dalirrooyfard.

A full version of this paper is available on the arXiv [2].

^{**} Parts of this work were carried out while M. Neuwöhner was affiliated with LSE.

‘uncrossing’, which has proven to be a powerful technique in graph orientations and combinatorial optimization (see [7], Chapters 48, 49, 55, 60, and [6], Part III).

Motivated by a conjecture on graph orientations by Chudnovsky, Edwards, Kim, Scott, and Seymour [4], which was recently proved by the authors [1], we introduce new notions and problems on crossing families, which are of independent interest. We then show an application to the recently proved conjecture.

A *cosigning* of a set family $F \subseteq 2^V \setminus \{\emptyset, V\}$ is a signing $\sigma : V \rightarrow \{+, -\}$ such that every set in F includes a *positive* element u ($\sigma(u) = +$) and excludes a *negative* element v ($\sigma(v) = -$). This notion naturally arises in digraphs over vertex set V where every vertex is a source or a sink, and every $U \in F$ receives an incoming arc. Our first result gives necessary and sufficient conditions for crossing families that admit a cosigning.

Theorem 1. *Let V be a finite set, and let $F \subseteq 2^V \setminus \{\emptyset, V\}$ be a crossing family. Then F admits a cosigning if and only if every set in F includes an element u and excludes an element v such that $V \setminus \{u\}$ and $\{v\}$ do not belong to F .*

The necessity is fairly clear: given a cosigning of *any* set family F , for every set U in the family, the condition holds for any positive element u inside U , and any negative element v outside. The crux of the theorem is in establishing the sufficiency of the condition for crossing families, which is done in Section 2.1; we shall outline our approach shortly after defining a stronger variant of cosignings.

Given a set family $F \subseteq 2^V \setminus \{\emptyset, V\}$, a $\cap\cup$ -closed³ *cosigning* is a signing $\sigma : V \rightarrow \{+, -\}$ such that $X \cap Y$ contains a positive element for all $X, Y \in F$ that intersect, and $X \cup Y$ excludes a negative element for all $X, Y \in F$ that *co-intersect*, i.e., $X \cup Y \neq V$. Observe that when $X = Y = U$ we recover the condition that defines a cosigning, so every $\cap\cup$ -closed cosigning is indeed a cosigning.

We give the following necessary and sufficient conditions for crossing families that admit $\cap\cup$ -closed cosignings. Necessity follows in the exact same manner as in Theorem 1 (for all set families).

Theorem 2. *Let V be a finite set, and let $F \subseteq 2^V \setminus \{\emptyset, V\}$ be a crossing family. Then F admits a $\cap\cup$ -closed cosigning if and only if for all $Z, T \in F$ that intersect, $Z \cap T$ includes an element u such that $V \setminus \{u\} \neq B \cup W$ for all $B, W \in F$ and for all $X, Y \in F$ that co-intersect, $X \cup Y$ excludes an element v such that $\{v\} \neq H \cap G$ for all $G, H \in F$.*

The proofs of sufficiency for both Theorems 1 and 2 are quite natural and algorithmic.

($\cap\cup$ -Closed) Cosigning Forcing Algorithm: Given a crossing family satisfying the necessary conditions, if there is an unsigned element u whose sign in a ($\cap\cup$ -closed) cosigning is ‘forced’ given the signed elements so far, then sign u accordingly; otherwise, pick an unsigned element and sign it arbitrarily; repeat.

³ read as cap-cup-closed or intersection-union-closed

The general descriptions of the algorithms above will be made more concrete in the proofs, which then amount to showing that at no iteration of the algorithms do we encounter a ‘conflict’, a scenario in which an unsigned element is forced simultaneously to be both positive and negative.

The Cosigning Forcing Algorithms are reminiscent of the Cameron-Edmonds algorithm for bicoloring the vertices of a balanced hypergraph such that no edge is monochromatic [3]. This was the first polynomial algorithm for bicoloring balanced matrices, and inspired the pursuit of the famous polynomial recognition algorithm for balanced hypergraphs nearly a decade later by Conforti, Cornuéjols, and Rao [5].

Letting $n := |V|$ and $m := |F|$, we prove that the Cosigning Forcing Algorithm has running time at most $m \cdot \binom{n+1}{2}$, while the $\cap\cup$ -closed variant has running time at most $2 \cdot \binom{m}{2} \cdot \binom{n+1}{2}$. Many crossing families that arise in applications, however, have size exponential in n , so it might be more appropriate to switch to a different computational model in which F is accessed via a suitable membership oracle. In this model, we prove that the Cosigning Forcing Algorithm can be executed in time at most $2n^3T$, where T is the best running time of an oracle-polynomial algorithm for computing the minimum of a modular function over a ‘well-provided lattice family’ (see Section 2.2). Finding a $\cap\cup$ -closed cosigning in oracle polynomial time is left as an open problem.

Our next contribution is to a covering problem on planar digraphs involving $\cap\cup$ -closed cosignings.

The Circle Problem. Let V be a finite set of vertices, and let σ be a $\cap\cup$ -closed cosigning for a crossing family $F \subseteq 2^V \setminus \{\emptyset, V\}$ without complementary sets. Thus, in the digraph $D = (V, A)$ with an arc from every negative vertex to every positive vertex, every set in F receives an incoming arc. Suppose now that the vertices in V are placed at distinct positions around a circle, in a fixed order, drawn on the plane. Can we then move to an ‘outer-planar’ subset of A such that every set in F still receives an incoming arc? Here, a set of arcs is *outer-planar* if the arcs can be drawn inside the circle in a planar manner. While this is not always possible, see Section 4.2 of the arXiv paper [2], we show that it can be guaranteed when every set in F forms an interval around the circle. We summarize this in the following theorem.

Theorem 3. *Let V be a finite set of vertices drawn around a circle, and let σ be a $\cap\cup$ -closed cosigning for a crossing family $F \subseteq 2^V \setminus \{\emptyset, V\}$ without complementary sets. Suppose every set in F forms an interval around the circle. Then there exists an outer-planar set of arcs from the negative to the positive vertices such that every set in F receives an incoming arc.*

None of the conditions on F in Theorem 3 can be dropped while keeping the others; see Section 4.2 of the arXiv paper [2] for more. For example, consider four vertices v_1, v_2, v_3, v_4 arranged clockwise around the circle, with $\sigma(v_1) = \sigma(v_4) = -$ and $\sigma(v_2) = \sigma(v_3) = +$. If $F = \{\{v_1, v_2\}, \{v_3, v_4\}\}$, no outer-planar arc set from negative to positive vertices can cover both sets of F . This set family

satisfies all conditions of Theorem 3 except that it has complementary sets. This theorem is useful in building planarity-preserving gadgets for digraphs. In particular, as our next contribution, we apply it to find disjoint dijoins in 0, 1-weighted planar digraphs.

Disjoint Dijoins. Let $D = (V, A)$ be a digraph. A *dicut* is a cut of the form $\delta^+(U) \subseteq A$ where $U \subset V, U \neq \emptyset$ and $\delta^-(U) = \emptyset$. We call U the *shore* of the dicut $\delta^+(U)$. The dicut shores of D form a crossing family of subsets of V . A blocking notion is that of a *dijoin*, which is an arc subset that intersects every dicut at least once.

Consider arc weights $w \in \{0, 1\}^A$ such that every dicut of D contains at least two weight-1 arcs. Suppose further that the weight-1 arcs form one connected component, not necessarily spanning all the vertices. Chudnovsky et al. [4] conjectured that the weight-1 arcs can then be decomposed into two dijoins of D . Recently, we proved this conjecture to be true by using integrality properties of the intersection of two submodular flow systems [1]. In this paper, we use graph-theoretic techniques to prove the conjecture for planar digraphs.

Theorem 4. *Let (D, w) be a 0, 1-(arc-)weighted planar digraph where every dicut has weight at least 2 and the weight-1 arcs form a connected component, not necessarily spanning. Then the weight-1 arcs contain two disjoint dijoins of D .*

Let us give a brief outline of the proof, in part motivating the circle problem. Chudnovsky et al. ([4], 1.6 part 2) proved Theorem 4 when the weight-1 arcs form a connected component that is *spanning*. This seemingly harmless condition turns out to be a key assumption in their proof which breaks down without it, and we do not see how to salvage it. Instead, we use their result in a black-box fashion.

Suppose the weight-1 arcs form a connected component that is not spanning, and let v be a *weight-0 vertex*, i.e., a vertex incident only to weight-0 arcs. Our goal then is to delete v and all the arcs incident with it, and to then install a planarity-preserving gadget connecting the in- and out-neighbors of v that maintains some key features of the problem. We repeat this until the weight-1 arcs form a spanning connected component, and then apply the Chudnovsky et al. result to prove Theorem 4.

Toward this end, let $N^-(v), N^+(v) \subset V$ be the sets of in- and out-neighbors of v , respectively, placed around a circle in the same order as they appear around v in the planar drawing of D . We then delete v and all the arcs incident with it, and add a new arc from every vertex in $N^-(v)$ to every vertex in $N^+(v)$, as shown in Figure 1. Although this reduction essentially maintains the dicuts and their weights, as well as the dijoins, it can make the digraph non-planar. We therefore have to remove some of the newly added arcs, but in doing so we potentially create new dicuts and thus restrict the family of dijoins. This turns out to be a nonissue if the minimum weight of a dicut remains at least 2.

We carefully choose the following crossing family for which $N^-(v), N^+(v)$ yields a $\cap\cup$ -closed cosigning:

$$\{U \subseteq V \setminus \{v\} \mid \delta^+(U) \text{ is a dicut of weight at most 1 in } D \setminus \{v\}\}$$

We prove in the arXiv paper [2], Section 5, that this family satisfies the hypotheses of Theorem 3. We then apply this theorem to decide which of the newly added arcs to keep to regain planarity.

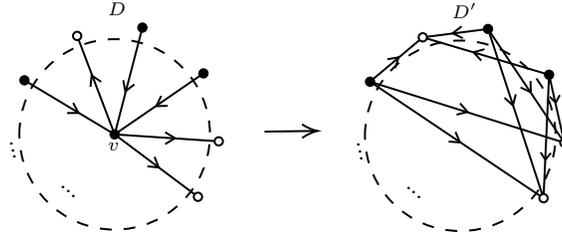


Fig. 1. (Left) A weight-0 vertex v and its in- and out-neighbors in a planar 0, 1-weighted digraph. (Right) A naive reduction that does not preserve planarity. Theorem 3 tells us how to retain an outer-planar subset of these arcs such that every dicut has weight at least 2. Filled-in and non-filled-in vertices are the in and out neighbors of v , respectively.

Outline of The Paper. Section 2 studies cosignings of a crossing family, describes the Cosigning Forcing Algorithm in detail, and proves Theorem 1. It also includes a brief commentary on the proof of Theorem 2. Section 3 outlines the algorithmic proof of Theorem 3, and comments on its polynomial running time. See the arXiv paper [2] for the skipped proofs, and for further discussions.

2 Cosigning a Crossing Family

In this section, we prove Theorem 1, providing necessary and sufficient conditions for when a crossing family admits a cosigning. As part of our proof, we devise a Cosigning Forcing Algorithm. We then analyze the algorithm’s running time in two computational models depending on how the crossing family is accessed.

2.1 The Cosigning Forcing Algorithm

Let V be a finite set, and let $F \subseteq 2^V \setminus \{\emptyset, V\}$ be a crossing family. We prove that F admits a cosigning if and only if every set in F includes an element u and excludes an element v such that $V \setminus \{u\}$ and $\{v\}$ do not belong to F .

We argued necessity in the introduction already. To show sufficiency, assume that the conditions hold. We shall apply the Cosigning Forcing Algorithm from the introduction to sign the elements one by one; let us define ‘forcings’ formally.

At any iteration of the algorithm, an unsigned element v is *forced to be positive* if there exists a set $U \in F$ such that $v \in U$ and all the elements in $U \setminus \{v\}$ are negatively signed; this forcing is *trivial* if $\{v\} \in F$.

Similarly, an unsigned element u is *forced to be negative* if there exists a set $W \in F$ such that $u \notin W$ and all the elements outside W other than u are positively signed; this forcing is *trivial* if $V \setminus \{u\} \in F$.

The algorithm starts by first signing all the trivially forced elements. We show that there is no element w that is trivially forced to be both positive and negative, i.e., a *conflict*, since otherwise $\{w\}, V \setminus \{w\} \in F$ thus contradicting our condition for either of the sets.

The algorithm then proceeds as follows. If there is an unsigned forced element, sign it accordingly — we prove that this can be done without a conflict. If not, pick an unsigned element, and sign it arbitrarily. Repeat until all elements are signed.

Suppose for a contradiction that the algorithm encounters a conflict. Take the first one, which is an element v that is forced to be both positive and negative. Then there exist sets $U, W \in F$ such that $v \in U$ with all the elements of $U \setminus \{v\}$ signed negatively, and $v \notin W$ with all the elements outside W other than v signed positively. In particular, all the elements in $U \cap W$ are negatively signed, and all the elements outside $U \cup W$ are positively signed. Observe that if one forcing is trivial, say $U = \{v\}$, then all signings so far have been trivially forced, in particular all the elements outside of W are trivially forced to be positive, contradicting our assumption for this set. Thus, v is not trivially forced, and so $|U|, |V \setminus W| \geq 2$. Let $u \in U \setminus \{v\}$, which is negatively signed, and therefore belongs to W . Let $z \in V \setminus (W \cup \{v\})$, which is positively signed, and is therefore outside U . Subsequently, $u \in U \cap W$ and $z \notin U \cup W$, so $U \cap W \neq \emptyset$ and $U \cup W \neq V$. Thus, as F is a crossing family, we have $U \cap W \in F$. However, all the elements in $U \cap W$ are negatively signed, and this indicates an earlier conflict.

More precisely, let w be the last element that the algorithm signs from $U \cap W$, which occurs at an earlier iteration of the algorithm. Then w must have been forced to be positive due to the remaining signs in $U \cap W \in F$. However, the algorithm decided to sign w negatively, a decision that must have been forced. Therefore, w was forced to both signs at an earlier iteration, thereby contradicting our choice of v .

We showed that the algorithm signs all the elements without a conflict. At the end, every set $U \in F$ must have a positive element inside. If not, the element l that was signed last in $U \in F$ must have been forced to be positive, yet the algorithm forced it to be negative, indicating a conflict at l , a contradiction. Similarly, every set in F must exclude a negatively signed element. Thus, the output is a cosigning, thereby finishing the proof. \square

2.2 Running Time Analysis

Given the setup of Theorem 1, the Cosigning Forcing Algorithm from the previous subsection takes time at most $m \cdot \binom{n+1}{2}$, where $n = |V|$ and $m = |F|$. In each iteration, we go through each set in F at most once to identify a suitable unsigned forced element, thus taking time at most mn' where n' is the number of unsigned elements. Subsequently, the total running time of the algorithm is at most $\sum_{n'=1}^n mn' = m \cdot \binom{n+1}{2}$.

Let us analyze the running time in a different computational model where V is given explicitly, but the crossing family F is accessed via a *membership oracle* which, given any $U \subseteq V$, tells us in unit time whether or not $U \in F$.

To avoid exponential information-theoretic lower bounds, we also assume that the crossing family F is *well-provided*. This means that for all distinct $u, v \in V$, the subfamily $F_{uv} := \{U \in F : u \in U, v \notin U\}$ is *well-provided*. Briefly speaking, F_{uv} is a lattice family, and can therefore be provided compactly by indicating its unique inclusion-wise minimal and maximal sets along with a pre-order on the elements — see [7], §49.3 for more details. To analyze the running time of the algorithm in this model, we need to describe a polynomial-time subroutine for deciding whether an unsigned element v is forced to a particular sign. This can be done by signing v the other way and then deciding whether the signing is *invalid*, i.e., there is a set in F whose elements inside are all negatively signed or whose elements outside are all positively signed.

Suppose we want to check whether an unsigned element u is forced to be positive. We sign u negatively and check whether there is a set $U \in F$ with $u \in U$ such that all elements in U are negatively signed. Consider the vector $a \in \{0, 1\}^V$ where $a_v = 0$ if v is negatively signed, and $a_v = 1$ otherwise. Let $f : 2^V \rightarrow \mathbb{Z}_{\geq 0}$ be the modular function defined as $f(U) = \sum_{v \in U} a_v$ for all $U \subseteq V$. Then $\min_{u \in U \in F} f(U) = 0$ if and only if there is a set in $U \in F$ with $u \in U$ whose elements inside are all negatively signed; in this case, any minimizer U^* will be such a set. To find it, we iterate over all $v \in V \setminus \{u\}$ and compute $\min_{U \in F_{uv}} f(U)$ using any algorithm for finding the minimum of a (sub)modular function over a lattice family, which can be done in oracle polynomial time (see [7], (49.25)). Since $\{V \setminus U : U \in F\}$ is a well-provided crossing family as well, we can check whether u is forced to be negative analogously. If we denote the best running time of an oracle polynomial time algorithm for finding the minimum of a (sub)modular function over a lattice family by T , we can find a forced element (or decide that no element is forced) in time at most $2 \cdot n \cdot (n - 1) \cdot T$. Over all n iterations, this yields a running time of at most $2n^3 \cdot T$ in the oracle model.

2.3 On $\cap\cup$ -Closed Cosignings

In order to handle $\cap\cup$ -closed cosignings, we generalize the notion of forcings in a natural way: An unsigned element v is *forced to be positive* if there exist $Z, T \in F$, possibly equal, such that $v \in Z \cap T$ and all the other elements in $Z \cap T$ are negatively signed; the forcing is *trivial* if $Z \cap T = \{v\}$. Similarly, an unsigned element v is *forced to be negative* if there exist $X, Y \in F$, possibly equal, such that $v \notin X \cup Y$ and all the other elements outside $X \cup Y$ are positively signed; the forcing is *trivial* if $X \cup Y = V \setminus \{v\}$.

As for cosignings, the conditions imposed immediately imply that no element can be trivially forced to both signs. We then show the algorithm never encounters a conflict similarly as in the proof of Theorem 1. Concerning the running time analysis, we are able to provide a polynomial guarantee only when the set family is provided explicitly.

3 The Circle Problem

In this section, we define the circle problem, which, roughly speaking, is concerned with the problem of covering a crossing family of circular intervals by an outer-planar set of arcs. More precisely, given a set of vertices V around a circle C , a crossing family F of circular intervals without complementary sets, and a $\cap\cup$ -closed cosigning σ , our goal will be to find an outer-planar set of arcs, oriented from the set V^- of negative vertices to the set V^+ of positive vertices, such that each interval in F receives an entering arc. We will do that algorithmically thus proving Theorem 3. Our algorithm proceeds as follows: Starting with an empty arc set, we iteratively add a suitable outer-planar set of arcs and delete all sets from the family F that receive an incoming arc, and repeat. This approach is motivated by the following lemma (proven in the arXiv paper [2], Section 4.4):

Lemma 1. *Let (V, σ, F) be an instance of the circle problem. Let E be a set of arcs from V^- to V^+ . Let F' consist of all the sets in F that are not covered by E . Then (V, σ, F') is a valid instance of the circle problem.*

The main obstacle in pursuing such a recursive strategy is how to choose the arc set E in such a way that the new instance (V, σ, F') admits a solution E' that is compatible with E in the sense that $E \cup E'$ remains outer-planar (and how to enforce this property without having to handle extra constraints in later iterations). To this end, we first observe that we can add arcs between adjacent vertices ‘for free’; these will never cross any other arcs. Adding such arcs and reducing the instance accordingly will be the first step of our algorithm. In general, we will of course not get away with only adding arcs between adjacent vertices and when adding an arc between non-adjacent vertices, this will separate the vertex sets into two parts, which can no longer be joined by any arcs. In the following, we will design a set of reduction rules that will, roughly speaking, add a set of arcs involving a small number of adjacent *sign-blocks* (see Section 3.1), delete all covered sets, and then restrict the instance by deleting all endpoints of the added arcs except for the two outer ones. This will ensure that further arcs that we add do not cross any previously added one. The difficulty is to ensure that the deletion of elements produces a valid instance.

3.1 Preliminaries and Notation

For a set of n vertices V around a circle C where each vertex is associated with a sign given by $\sigma : V \rightarrow \{+, -\}$, and a family F of subsets of V , we say (V, σ, F) is a (valid) instance of the circle problem if F has the following properties:

- P0** For every $U \in F$, U is an interval of consecutive vertices around C .
- P1** F is a crossing family.
- P2** For every $U \in F$, there exists a positive vertex in U and there exists a negative vertex in $V \setminus U$.

P3 For every $U, W \in F$ such that $U \cap W = \emptyset$, there exists a negative vertex in $V \setminus (U \cup W)$.

P4 For every $U, W \in F$ such that $U \cup W = V$, there exists a positive vertex in $U \cap W$.

Observe that properties **P2-P4** are equivalent to stating that σ is a $\cap\cup$ -closed cosigning and F does not contain a pair of complementary sets. Section 4.2 of the arXiv paper [2] provides an example and explains why each property is necessary for Theorem 3.

Let (V, σ, F) be an instance of the circle problem. Denote by V^+ and V^- the set of vertices of V with $+$ and $-$ signs, respectively.

For any set $U \subseteq V$, let $U^c = V \setminus U$. Denote by U^+ the set of $+$ -signed vertices in U and by U^- the set of $-$ -signed vertices in U . Furthermore, let U^{c+} be the set of $+$ -signed vertices in U^c and let U^{c-} be the set of $-$ -signed vertices in U^c .

For any $V' \subseteq V$ and $U \subseteq V$, let $U[V'] = U \cap V'$. We define the restriction of F to the ground set V' to be $F[V'] = \{U[V'] \mid U \in F, U[V'] \neq \emptyset\}$. Furthermore, denote by $\sigma[V']$ the restriction of the sign function σ to V' .

Let (m, p) be an arc from a $-$ -signed vertex m to a $+$ -signed vertex p . We say (m, p) covers $U \in F$ if $m \notin U$ and $p \in U$. For a set of arcs A from V^- to V^+ , let F_A consist of all the sets $W \in F$ such that W is covered by some arc in A . We say a set A of arcs covers F if $F_A = F$.

For a set $U \in F$, a vertex $u \in U$ is an *end vertex* if it is at the end of the interval U forms on the circle. If $|U| = 1$ then it has one end vertex and if $|U| \geq 2$, then it has two end vertices. If v is an end vertex of U , an *adjacent out-neighbor* of v with respect to U is a vertex v' next to v on the circle with $v' \notin U$. Note that by **P2**, for any $U \in F$, we have $U \neq V$ and hence its end vertices have adjacent out-neighbors.

We call a set of consecutive vertices with the same sign around the circle a *block*. We say a set in $U \in F$ is a *1-block* if it forms one block around the circle, i.e., all of its vertices have the same sign. Note that by **P2**, the common sign is a $+$. We call a set $W \in F$ a *2-block* if W^+ and W^- are two nonempty blocks. Moreover, we say a set $X \in F$ is a *co-2-block* if X^{c+} and X^{c-} are two consecutive nonempty blocks. For a vertex v , the *sign-block* of v is the maximal block of vertices around the circle that contains v .

We call a set of directed arcs A from V^- to V^+ *outer-planar* if we can draw the arcs in A inside the circle C with no two arcs crossing each other. For an instance of the circle problem (V, σ, F) , we say that a set of directed arcs E from V^- to V^+ is a *solution* to (V, σ, F) if E is outer-planar and covers F .

For a given instance of the circle problem (V, σ, F) , we define its *dual* $(V, \bar{\sigma}, \bar{F})$ as follows: Let $\bar{\sigma}$ be the flipped signing, i.e., $\bar{\sigma}(v) = +$ if and only if $\sigma(v) = -$ and vice versa. Furthermore, let $\bar{F} = \{V \setminus U \mid U \in F\}$.

Remark 1. For a given instance of the circle problem (V, σ, F) , its dual $(V, \bar{\sigma}, \bar{F})$ is an instance of the circle problem as well, i.e., \bar{F} has properties **P0-P4**. Furthermore, if E is a solution to (V, σ, F) , then the arc set \bar{E} that we obtain from E by reversing the direction of each arc, is a solution to $(V, \bar{\sigma}, \bar{F})$.

3.2 Algorithm

In the following, we provide a full overview of our algorithm to solve the circle problem. This algorithm can be executed in $O(n^2m^2)$ where $n = |V|$ and $m = |F|$. The lemmas referenced throughout the algorithm can be found in the arXiv paper [2], wherein further details and the proof of Theorem 3 are provided.

Algorithm

Input: an instance of the circle problem (V, σ, F) around a circle C

Output: an outer-planar arc subset E from V^- to V^+ that covers F

1. Let E be the set of all the arcs between adjacent vertices with opposite signs, oriented from the $-$ signed vertex to the $+$ signed one. Moreover, let $F_1 \subseteq F$ be the sets not covered by E and let $V_1 = V$ (Lemma 4.5).
2. While there exists $W^* \in F_1$ with a $-$ signed end-vertex whose adjacent out-neighbor has a $+$ sign:
 - (a) If W^* is not a co-2-block, let m be its $-$ signed end-vertex with $+$ signed adjacent out-neighbor p_1 . Let p_1, p_2, \dots, p_t for some $t \geq 1$ be the vertices of the sign-block of p_1 in this order and let m' be the $-$ signed vertex adjacent to p_t (Lemma 4.12 and figure 2.a). Then update: $E \leftarrow E \cup \{(m, p_i) \mid i \in [t]\} \cup \{(m', p_t)\}$, $F_1 \leftarrow \{U \in F_1 \mid U \text{ not covered by } E\}$, $V_1 \leftarrow V_1 \setminus \{p_1, p_2, \dots, p_t\}$ and $F_1 \leftarrow F_1[V_1]$.
 - (b) If W^* is a co-2-block, let m_1 be its $-$ signed end-vertex with $+$ signed adjacent out-neighbor p . Let m_1, m_2, \dots, m_ℓ for some $\ell \geq 1$ be the vertices of the sign-block of m_1 in this order and let p' be the $+$ signed vertex adjacent to m_ℓ (Lemma 4.13 and figure 2.b). Then update: $E \leftarrow E \cup \{(m_i, p) \mid i \in [\ell]\} \cup \{(m_\ell, p')\}$, $F_1 \leftarrow \{U \in F_1 \mid U \text{ not covered by } E\}$, $V_1 \leftarrow V_1 \setminus \{m_1, m_2, \dots, m_\ell\}$ and $F_1 \leftarrow F_1[V_1]$.
3. If consecutive vertices $m_2, m_1, p_1, p_2 \in V_1$ with this order on C satisfy $\sigma(m_1) = \sigma(m_2) = -$ and $\sigma(p_1) = \sigma(p_2) = +$, there are three cases:
 - (a) One of m_1 or p_1 is *removable*, i.e., either there are no sets $U, W \in F_1$, possibly equal, such that m_1 is the only negative vertex of $V_1 \setminus (U \cup W)$, or there are no sets $U, W \in F_1$, possibly equal, such that p_1 is the only positive vertex of $U \cap W$. Let $u \in \{m_1, p_1\}$ be the removable vertex. Update $V_1 \leftarrow V_1 \setminus \{u\}$ and $F_1 \leftarrow F_1[V_1]$. Add a new arc to E between the new adjacent opposite signed vertices (m_2, p_1 if $u = m_1$ or m_1, p_2 if $u = p_1$) and remove the covered sets from F_1 (Lemmas 4.20, 4.21 and 4.22).
 - (b) $\{m_1, p_1\} \in F_1$, and there exist disjoint sets $U, W \in F_1$ where m_1 is the only $-$ signed vertex in $V_1 \setminus (U \cup W)$, let T be the set of all such pairs (U, W) . For each pair exactly one set, say W , excludes m_2 (Lemma 4.23). Let p_1, p_2, \dots, p_t for some $t \geq 2$ be the vertices of the sign-block of p_1 in this order. Then some $i \in \{2, \dots, t\}$ satisfies $p_i \in W$ and $p_{i-1} \notin W$ (Lemma 4.23). Denote that as $f(U, W) = i$. Choose the pair maximizing $f(U, W)$ with value j . See figure 2.c. Then update: $E \leftarrow E \cup \{(m_2, p_k) \mid k \in [j]\} \cup \{(m_1, p_1)\}$, $F_1 \leftarrow \{U \in F_1 \mid U \text{ not covered by } E\}$, $V_1 \leftarrow V_1 \setminus \{m_1, p_1, p_2, \dots, p_{j-1}\}$, and $F_1 \leftarrow F_1[V_1]$.

- (c) $V_1 \setminus \{m_1, p_1\} \in F$, and there exists sets $U, W \in F_1$ such that $U \cup W = V_1$ and p_1 is the only + signed vertex in $U \cap W$, let T be the set of all such pairs (U, W) . For each pair exactly one set, say W contains p_2 (Lemma 4.24). Let $m_\ell, m_{\ell-1}, \dots, m_2, m_1$ for some $\ell \geq 2$ be the vertices of the sign-block of m_1 in this order. Then some $i \in \{2, \dots, \ell\}$ satisfies $m_{i-1} \in W$ and $m_i \notin W$ (Lemma 4.24). Denote that as $f(U, W) = i$. Choose the pair minimizing $f(U, W)$ with value j . See figure 2.d. Then update: $E \leftarrow E \cup \{(m_k, p_2) | k \in [j]\} \cup \{(m_1, p_1)\}$, $F_1 \leftarrow \{U \in F_1 | U \text{ not covered by } E\}$, $V_1 \leftarrow V_1 \setminus \{p_1, m_1, m_2, \dots, m_{j-1}\}$ and $F_1 \leftarrow F_1[V_1]$.

Go to Step 2.

4. If consecutive vertices $m_2, m_1, p, m_3 \in V_1$ with this order on C satisfy $\sigma(m_1) = \sigma(m_2) = \sigma(m_3) = -$ and $\sigma(p) = +$, then m_1 is removable (Lemma 4.26). Update $V_1 \leftarrow V_1 \setminus \{m_1\}$, $E \leftarrow E \cup \{(m_2, p)\}$, $F_1 \leftarrow F_1[V_1]$ and remove the covered sets from F_1 . Go to Step 2.
5. If consecutive vertices $p_2, p_1, m, p_3 \in V_1$ with this order on C satisfy $\sigma(p_1) = \sigma(p_2) = \sigma(p_3) = +$ and $\sigma(m) = -$, then p_1 is removable (Lemma 4.27). Update $V_1 \leftarrow V_1 \setminus \{p_1\}$, $E \leftarrow E \cup \{(m, p_2)\}$, $F_1 \leftarrow F_1[V_1]$ and remove the covered sets from F_1 . Go to Step 2.
6. Output E .

4 Future Research Directions

In this paper, we characterized when crossing families admit a $(\cap \cup)$ -closed cosigning. An interesting future research direction would be to characterize when such cosignings exist for other classes of set families. Also of interest is further applicability of such cosignings.

We provided polynomial algorithms for finding a $(\cap \cup)$ -closed cosigning when the crossing family is given explicitly. In combinatorial optimization, however, many set families have exponential size. We addressed this by providing an oracle polynomial algorithm for cosigning a well-provided crossing family. A very interesting research direction is to develop such an algorithm for $\cap \cup$ -closed cosignings, or argue NP-hardness.

Finally, we built an outer-planar gadget for a digraph whose vertices, comprised of sources and sinks, are placed around a circle, and every cut coming from a certain cosigned crossing family is covered by an incoming arc. We used this to give a graph-theoretic proof of Theorem 4. Using techniques from submodular flows and polyhedral integrality, the authors have recently shown that this theorem extends to non-planar digraphs as well [1]. Another research direction is to find a graph-theoretic proof of this.

Acknowledgments. This work was supported in part by EPSRC grant EP/X030989/1.

Data Availability Statement. No data are associated with this article. Data sharing is not applicable to this article.

Disclosure of Interests. The authors have no competing interests to declare that are relevant to the content of this article.

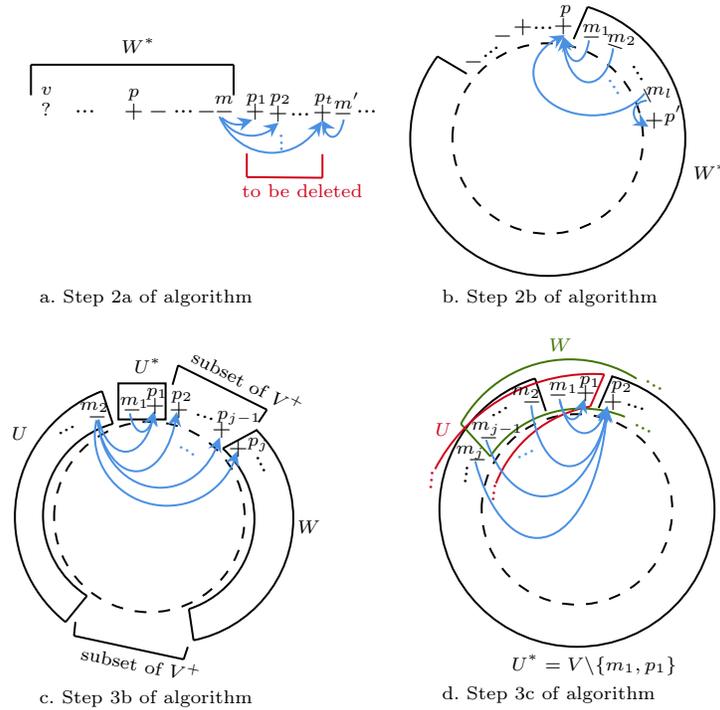


Fig. 2. Steps of the Algorithm for Theorem 3.

References

1. Abdi, A., Dalirrooyfard, M., Neuwohner, M.: Strong orientation of a connected graph for a crossing family. *Operations Research Letters* **62**, 107333, (2025). <https://doi.org/10.1016/j.orl.2025.107333>
2. Abdi, A., Dalirrooyfard, M., Neuwohner, M.: Cosigning Crossing Families and Outer-Planar Gadgets. *arXiv:2602.24124* (2026). <https://arxiv.org/abs/2602.24124>
3. Cameron, K., Edmonds, J.: Existentially polytime theorems. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* **1**, 83–100 (1990)
4. Chudnovsky, M., Edwards, K., Kim, R., Scott, A., Seymour, P.: Disjoint dijoins. *Journal of Combinatorial Theory, Series B* **120**, 18–35 (2016). <https://doi.org/10.1016/j.jctb.2016.04.002>
5. Conforti, M., Cornuéjols, G., Rao, M.R.: Decomposition of balanced matrices. *Journal of Combinatorial Theory, Series B* **77**, 292–406 (1999). <https://doi.org/10.1006/jctb.1999.1932>
6. Frank, A.: *Connections in Combinatorial Optimization*. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, Oxford (2011)
7. Schrijver, A.: *Combinatorial Optimization. Polyhedra and Efficiency*. Springer, Berlin, Heidelberg (2003)