Arc connectivity and submodular flows in digraphs

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Abstract

Let $D = (V, A)$ be a digraph. For an integer $k \geq 1$, a *k-arc-connected flip* is an arc subset of D such that after reversing the arcs in it the digraph becomes (strongly) k-arc-connected. The main result of this paper introduces a sufficient condition for the existence of a k-arc-connected flip that is also a submodular flow for a crossing submodular function.

The main result has several applications to Graph Orientations and Combinatorial Optimization, including the following two highlights. Suppose the underlying graph of D is τ-edge-connected for some integer $\tau \geq 2$. One consequence is that for any crossing submodular function $f : \mathcal{C} \subseteq 2^V \to \mathbb{Z}$ such that $f(U) \geq \frac{1}{2}(|\delta^+(U)| - |\delta^-(U)|)\ \forall U \in \mathcal{C}$, there exists a $\lfloor \tau/2 \rfloor$-arc-connected flip whose incidence vector is also an f-submodular flow. This strengthens Nash-Williams' so-called *weak orientation theorem* and its near-Eulerian extension. A second consequence is that A can be decomposed into a k-arc-connected flip and a $(\tau - k)$-*dijoin* for any $k \in \{1, \ldots, \lfloor \tau/2 \rfloor\}$. This provides evidence for Woodall's conjecture on such digraphs.

The main result follows from a surprising sufficient condition for the existence of capacitated integral solutions to the intersection of two submodular flow systems. This result implies the classical result of Edmonds and Giles on the box-total dual integrality of a submodular flow system. It also has the consequence that in a weakly connected digraph, the intersection of two submodular flow systems is totally dual integral.

Keywords: graph orientation, k-arc-connected flip, weak orientation theorem, Woodall's conjecture, submodular flows, total dual integrality

1 Introduction

Graph orientation is a rich area of graph theory. The basic problem consists in orienting the edges of an undirected graph in order to obtain a k-arc-connected digraph, and giving conditions under which finding such an orientation is possible. Various restrictions on the orientation can be imposed, leading to an extensive literature on graph orientation, see [6] for an example, and ([7], Chapter 9) and ([17], Chapter 61) for further results and references. Equivalently, one can start from a digraph and flip the orientation of some of the arcs in order to obtain desired connectivity properties.

Let $D = (V, A)$ be a digraph. For $U \subseteq V$ denote by $\delta^+_D(U)$ and $\delta^-_D(U)$ the sets of arcs leaving and entering U, respectively. We shall drop the subscript D whenever it is clear from the context.

Definition 1. *For an integer $k \geq 1$, a* k-arc-connected flip of $D = (V, A)$ *is a subset $J \subseteq A$ such that after flipping the arcs of J the digraph becomes k-arc-connected, that is, $|\delta^+(U) \cap J| + |\delta^-(U)| - |\delta^-(U) \cap J| \geq k$ for all $U \subsetneq V, U \neq \emptyset$, or equivalently, $|\delta^+(U) \cap J| - |\delta^-(U) \cap J| \leq |\delta^+(U)| - k$ for all $U \subsetneq V, U \neq \emptyset$.*

An important result is Nash-Williams' *weak orientation theorem*, stating that if the underlying graph is $2k$-edge-connected, then D has a k-arc-connected flip ([14], see [6]). Our main theorem strengthens this result in two directions. To state it we need to borrow a few notions from Submodular Optimization.

Let $\mathcal{C}$ be a *crossing family* over a finite ground set V. That is, $\mathcal{C}$ is a family of subsets of V such that for all $U, W \in \mathcal{C}$ with $U \cap W \neq \emptyset, U \cup W \neq V$, we have $U \cap W, U \cup W \in \mathcal{C}$. A function $f : \mathcal{C} \to \mathbb{Z}$ is *crossing submodular* over $\mathcal{C}$ if, for all $U, W \in \mathcal{C}$ such that $U \cap W \neq \emptyset, U \cup W \neq V$, we have $f(U \cap W) + f(U \cup W) \leq f(U) + f(W)$. If we have $\geq$ or $=$ instead, then f is a crossing *supermodular* or *modular* function, respectively. For instance, $\{U \subsetneq V : U \neq \emptyset\}$ is a crossing family, and $U \mapsto |\delta^+_D(U)|$ is a crossing submodular function defined over the family. Another important example of a crossing family is $\{U \subsetneq V : \delta^-_D(U) = \emptyset, U \neq \emptyset\}$ over which $U \to |\delta^+_D(U)|$ is a crossing modular function.

The linear system $y(\delta^+(U)) - y(\delta^-(U)) \leq f(U)$ for all $U \in \mathcal{C}$ is a *submodular flow system*, and any feasible solution is called a *submodular flow*. Observe that the incidence vector of a k-arc-connected flip is a submodular flow for the crossing submodular function $U \mapsto |\delta^+(U)| - k$ defined on $\{U \subsetneq V : U \neq \emptyset\}$.

The following is the main result of the paper, which introduces a sufficient condition for the existence of a k-arc-connected flip whose incidence vector is also a submodular flow for another crossing submodular function.

Theorem 2. *Let $\tau, k \geq 1$ be integers. Let $D = (V, A)$ be a digraph where $|\delta^+(U)| + (\frac{\tau}{k} - 1)|\delta^-(U)| \geq \tau$ for all $U \subsetneq V, U \neq \emptyset$. Let $\mathcal{C}$ be a crossing family over ground set V, and let $f : \mathcal{C} \to \mathbb{Z}$ be a crossing submodular function such that $f(U) \geq \frac{k}{\tau}(|\delta^+(U)| - |\delta^-(U)|)$ for all $U \in \mathcal{C}$. Then D has a k-arc-connected flip J such that $f(U) \geq |J \cap \delta^+(U)| - |J \cap \delta^-(U)|$ for all $U \in \mathcal{C}$.*

We shall prove this theorem in §3. In that section, we see that for $\tau = 2k$ this result strengthens the weak orientation theorem and its near-Eulerian sharpening, and for $k = 1$ it reduces to a recent result on decomposing A into a *dijoin* and a $(\tau - 1)$-*dijoin* [1]. Furthermore, the theorem has several more applications: an extension of the weak orientation theorem in a different direction than above, a weaker version of *Woodall's conjecture* for τ-edge-connected digraphs [19], and a theorem on disjoint dijoins in $0, 1$-weighted digraphs [4].

We prove Theorem 2 by utilizing a surprising result on submodular flows. To explain it, we need a few notions from Integer Programming. Given $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, the linear system $Ax \leq b$ is *totally dual integral (TDI)* if for each $w \in \mathbb{Z}^n$, the dual of the linear program $\max\{w^\top x : Ax \leq b\}$ has an integral optimal solution whenever the LP admits an optimum [5]. The system $Ax \leq b$ is *box-TDI* if the system $Ax \leq b, \ell \leq x \leq u$ is TDI, for all $\ell, u \in \mathbb{Z}^n$ such that $\ell \leq u$. An important result is that if $Ax \leq b$ is TDI and $b \in \mathbb{Z}^m$, then $\{x : Ax \leq b\}$ is an *integral* polyhedron, that is, every non-empty face of it contains an integral point [10, 5]. In particular, if $Ax \leq b$ is box-TDI and $b \in \mathbb{Z}^m$, then $\{x : Ax \leq b, \ell \leq x \leq u\} = \{x : Ax \leq b\} \cap [\ell, u]$ is an integral polyhedron for all $\ell, u \in \mathbb{Z}^n$ such that $\ell \leq u$, that is, $\{x : Ax \leq b\}$ is a *box-integral polyhedron*.

A classical result of Edmonds and Giles states that a submodular flow system is box-TDI [5]. This important theorem laid the basis for beautiful min-max theorems and powerful polynomial and strongly polynomial algorithms for submodular flows. For in-depth surveys see [16, 8] and for a more recent treatment we recommend ([17], Chapter 60) and ([7], Chapter 16). In contrast, given two crossing submodular functions $f_i : \mathcal{C}_i \to \mathbb{Z}, i = 1, 2$ defined over (possibly different) crossing families $\mathcal{C}_i, i = 1, 2$ over the same ground set, it is folklore that the combined system $y(\delta^+(U)) - y(\delta^-(U)) \leq f_i(U)\;\forall\, U \in \mathcal{C},\; i = 1, 2$, is not box-TDI (and not even integral), and that finding a nonnegative integral solution to this system is NP-hard in general; see §A in the appendix for details.

Against this backdrop, Theorem 2 is all the more surprising as it provides ultimately a $0, 1$ solution to the intersection of two submodular flow systems. The theorem is a consequence of the following sufficient condition for the existence of capacitated integral solutions to the intersection of two submodular flow systems.

Theorem 3. *Let $D = (V, A)$ be a digraph. For $i = 1, 2$, let $\mathcal{C}_i$ be a crossing family over ground set V and $f_i : \mathcal{C}_i \to \mathbb{Z}$ a crossing submodular function, where $\min_{i=1,2} f_i(U) \leq 0$ for all $U \subsetneq V, U \neq \emptyset$ such that $\delta^+(U) = \delta^-(U) = \emptyset$.*[1] *Let*

$$P := \left\{ y \in \mathbb{R}^A : y(\delta^+(U)) - y(\delta^-(U)) \leq f_i(U) \quad \forall U \in \mathcal{C}_i,\, i = 1, 2 \right\}. \tag{1}$$

Then the following statements hold:

a. The system (1) *defining P is TDI. In particular, P is an integral polyhedron.*

b. Take $\ell \in (\mathbb{Z} \cup \{-\infty\})^A$, $u \in (\mathbb{Z} \cup \{+\infty\})^A$ such that $\ell \leq u$ and the following cut condition *is satisfied:*

$$\min_{i=1,2} f_i(U) \leq u(\delta^+(U)) - \ell(\delta^-(U)) \quad \forall U \subsetneq V,\, U \neq \emptyset. \tag{2}$$

Then every non-empty face of P contains $y^\star \in \mathbb{Z}^A$ satisfying $\ell \leq y^\star \leq u$.

We prove this theorem in §2, where we also present some more applications besides Theorem 2, including the original result of Edmonds and Giles. As a final comment of this section, we note that it is not necessarily true that P as in (1) is box-integral. In fact, $P \cap [\ell, u]$ is not necessarily integral even if ℓ, u satisfies the cut condition (2); for an example see §B of the appendix.

2 Intersection of two submodular flow systems

To prove Theorem 3 we need two ingredients from Submodular Optimization and Network Flows.

[1] We follow the convention that $f_i(U) = +\infty$ if $U \notin \mathcal{C}_i$.

First we need the following result, essentially stating that the intersection of two base systems is box-TDI.

Theorem 4 (see [17], Theorem 49.8, also [7], §14.4)**.** *For $i = 1, 2$, let $\mathcal{C}_i$ be a crossing family over ground set V, let $f_i : \mathcal{C}_i \to \mathbb{Z}$ be a crossing submodular function, and let k be an integer. Then the system $x(V) = k$; $x(U) \le f_1(U)\ \forall U \in \mathcal{C}_1$; $x(U) \le f_2(U)\ \forall U \in \mathcal{C}_2$ is box-TDI, and therefore defines a box-integral polyhedron.*

Given a digraph $D = (V, A)$ and $b \in \mathbb{R}^V$ such that $\mathbf{1}^\top b = 0$, a *b-transshipment* is a vector $y \in \mathbb{R}^A$ such that $y(\delta^+(v)) - y(\delta^-(v)) = b_v$ for every $v \in V$. We also need the following well-known result.

Theorem 5 (see [17], Corollary 11.2f)**.** *Let $D = (V, A)$ be a digraph. Take $b \in \mathbb{Z}^V$, $\ell \in (\mathbb{Z} \cup \{-\infty\})^A$, and $u \in (\mathbb{Z} \cup \{+\infty\})^A$ such that $\mathbf{1}^\top b = 0$ and $\ell \le u$. Then there exists a b-transshipment $y \in \mathbb{Z}^A$ such that $\ell \le y \le u$ if, and only if, $b(U) \le u(\delta^+(U)) - \ell(\delta^-(U))$ for all $U \subsetneq V, U \neq \emptyset$.*

We are now ready to prove Theorem 3.

Proof of Theorem 3. Choose $c \in \mathbb{Z}^A$ such that $\max\{c^\top y : y \in P\}$ admits an optimal solution, let $\omega^\star$ be the optimal value, and let F be the optimal face. Write $F = P \cap \{y : y(\delta^+(U)) - y(\delta^-(U)) = f_i(U)\ \forall U \in \mathcal{D}_i, i = 1, 2\}$ for some subfamily $\mathcal{D}_i$ of $\mathcal{C}_i$, $i = 1, 2$. Define the polyhedron $\widetilde{P} := \{x \in \mathbb{R}^V : \mathbf{1}^\top x = 0, x(U) \le f_i(U)\ \forall U \in \mathcal{C}_i, i = 1, 2\}$, and the face $\widetilde{F} := \widetilde{P} \cap \{x : x(U) = f_i(U)\ \forall U \in \mathcal{D}_i, i = 1, 2\}$.

Claim 1. *Choose $x \in \mathbb{R}^V, y \in \mathbb{R}^A$ such that y is an x-transshipment. Then $y \in P$ if and only if $x \in \widetilde{P}$, and $y \in F$ if and only if $x \in \widetilde{F}$.*

Proof of Claim. Since an x-transshipment exists, $\mathbf{1}^\top x = 0$ is automatically satisfied. The claim now follows from the equality $y(\delta^+(U)) - y(\delta^-(U)) = x(U)$ for all $U \subseteq V$. $\Diamond$

Pick an arbitrary point $\bar{y} \in F$, and define $\bar{x} \in \mathbb{R}^V$ by $\bar{x}_v := \bar{y}(\delta^+(v)) - \bar{y}(\delta^-(v))\ \forall v \in V$. By Claim 1, $\bar{x} \in \widetilde{F}$, so $\widetilde{F} \neq \emptyset$.

(b) First we prove the second part as its proof is shorter and contains the crux of the argument. It suffices to find an integral point $y^\star \in F$ satisfying $\ell \le y^\star \le u$. By Theorem 4, $\widetilde{P}$ is an integral polyhedron, hence $\widetilde{F}$ contains an integral point b. By the cut condition (2), we have that $b(U) \leq u(\delta^+(U)) - \ell(\delta^-(U))$ for all $U \subsetneq V, U \neq \emptyset$. Thus, by Theorem 5, there exists a b-transshipment $y^\star \in \mathbb{Z}^A$ such that $\ell \leq y^\star \leq u$. Since $b \in \widetilde{F}$, it follows from Claim 1 that $y^\star \in F$. This is the desired point.

(a) To prove this part it suffices to show that the dual of $\max\{c^\top y : y \in P\}$ has an integral optimal solution. Let $M \in \{0, \pm 1\}^{V \times A}$ denote the node-arc incidence matrix of D. It is well-known that M is a totally unimodular matrix. Observe that $\bar{x} = M\bar{y}$.

Claim 2. *There exists $w \in \mathbb{Z}^V$ such that $w^\top M = c^\top$, and $w^\top \bar{x} = \omega^\star$.*

Proof of Claim. Observe that whenever $y \in \mathbb{R}^A$ satisfies $My = \bar{x}$, then $y \in F$ by Claim 1, so $c^\top y = \omega^\star$. Thus, the linear system $My = \bar{x}, y \in \mathbb{R}^A$ implies the equation $c^\top y = \omega^\star, y \in \mathbb{R}^A$. Basic linear algebra dictates the existence of $w \in \mathbb{R}^V$ such that $w^\top M = c^\top$. Note that $w^\top \bar{x} = \omega^\star$. Since M is totally unimodular, and $c \in \mathbb{Z}^A$, we may choose $w \in \mathbb{Z}^V$. $\Diamond$

Claim 3. $\max\{w^\top x : x \in \widetilde{P}\} = \omega^\star$.

Proof of Claim. $(\geq)$ follows from $w^\top \bar{x} = \omega^\star$. $(\leq)$ Let $x' \in \widetilde{P}$. There exists an x'-transshipment. To show this, it suffices to prove that $x'(U) \leq 0$ whenever $\delta^+(U) = \delta^-(U) = \emptyset$, by Theorem 5. This holds because for any such U, $x'(U) \leq \min_{i=1,2} f_i(U) \leq 0$, where the last inequality holds by the hypothesis. Now let $y' \in \mathbb{R}^A$ be an x'-transshipment, that is, $My' = x'$. Observe that $y' \in P$ by Claim 1. Thus, $w^\top x' = w^\top My' = c^\top y' \leq \omega^\star$, where the last inequality follows from the definition of $\omega^\star$ and the fact that $y' \in P$. $\Diamond$

Consider now the dual of $\max\{w^\top x : x \in \widetilde{P}\}$:

$$\begin{array}{lrcll} \min & \sum_{i=1,2} \sum_{U \in \mathcal{C}_i} f_i(U) z_U^i & & & \\ \text{s.t.} & \sum_{i=1,2} \sum_{U \in \mathcal{C}_i} \chi^U z_U^i + \mathbf{1}\mu & = & w & \\ & z_U^i & \geq & 0 & U \in \mathcal{C}_i,\ i = 1, 2, \end{array} \tag{3}$$

where $\mu \in \mathbb{R}$ is the dual variable corresponding to $\mathbf{1}^\top x = 0$. By Theorem 4, the system of constraints of $\widetilde{P}$ is TDI, so because w is integral, it follows that (3) has an integral optimal solution $(\bar{z}, \bar{\mu})$.

Claim 4. $\bar{z} = (\bar{z}^i_U)_{U\in\mathcal{C}_i,\, i=1,2}$ *is an optimal solution to the dual of* $\max\{c^\top y : y \in P\}$*:*

$$\begin{array}{lrcl} \min & \sum_{i=1,2}\sum_{U\in\mathcal{C}_i} f_i(U) z^i_U & & \\ \textit{s.t.} & \sum_{i=1,2}\sum_{U\in\mathcal{C}_i}\left(\chi^{\delta^+(U)} - \chi^{\delta^-(U)}\right) z^i_U & = & c \\ & z^i_U & \geq & 0 \quad U \in \mathcal{C}_i,\, i = 1,2, \end{array} \tag{4}$$

Proof of Claim. By Claim 3 and LP Strong Duality, (4) has optimal value $\omega^\star$. Obviously $\bar{z}$ has objective value $\omega^\star$, hence we only need to show feasibility. This follows immediately from the fact that $M^\top w = c$, $M^\top \chi^U = \chi^{\delta^+(U)} - \chi^{\delta^-(U)}$ for every $U \subsetneq V$, $U \neq \emptyset$, and $M^\top \mathbf{1} = 0$, because

$$\sum_{i=1,2}\sum_{U\in\mathcal{C}_i}\left(\chi^{\delta^+(U)} - \chi^{\delta^-(U)}\right)\bar{z}^i_U = M^\top\left(\sum_{i=1,2}\sum_{U\in\mathcal{C}_i}\chi^U \bar{z}^i_U + \mathbf{1}\bar{\mu}\right) = M^\top w = c.$$

$\Diamond$

Claim 4 finishes the proof of the first part. $\square$

Theorem 3 implies the classical theorem of Edmonds and Giles [5]. One way to prove it is to use a recent characterization of box-TDI systems. Consider a polyhedron $Q := \{y : Ay \leq b\}$. For an integer $k \geq 1$, the k^{th} dilation of Q is $kQ := \{y : Ay \leq kb\}$. Q is *principally box-integral* if kQ is box-integral for all integers $k \geq 1$ such that kQ is integral. This notion was coined recently by Chervet, Grappe, and Robert who proved that $Ax \leq b$ is box-TDI if, and only if, $Ax \leq b$ is TDI and Q is principally box-integral [3].

Theorem 6 **([5]).** *Let $D = (V, A)$ be a digraph, $\mathcal{C}$ a crossing family over ground set V, and $f : \mathcal{C} \to \mathbb{Z}$ a crossing submodular function. Then $y(\delta^+(U)) - y(\delta^-(U)) \leq f(U)\ \forall U \in \mathcal{C}$ is box-TDI.*

Proof. We need the following two claims.

Claim 1. $y(\delta^+(U)) - y(\delta^-(U)) \leq f(U)\ \forall U \in \mathcal{C}$ *is TDI.*

Proof of Claim. Let $\mathcal{C}_1 := \mathcal{C}$ and $f_1 := f$. Let $\mathcal{C}_2 := \{U \subsetneq V : U \neq \emptyset, \delta^+(U) = \delta^-(U) = \emptyset\}$ and $f_2(U) := 0$ for all $U \in \mathcal{C}_2$. Clearly, $\mathcal{C}_2$ is a crossing family, and f_2 is a crossing submodular function defined over $\mathcal{C}_2$. Moreover, $\min_{i=1,2} f_i(U) \leq 0$ for all $U \subsetneq V, U \neq \emptyset$ such that $\delta^+(U) = \delta^-(U) = \emptyset$. Subsequently, it follows from Theorem 3 (a) that the combined system $y(\delta^+(U)) - y(\delta^-(U)) \leq$

$f_i(U)\ \forall U \in \mathcal{C}_i, i=1,2$ is TDI. However, this system is precisely $y(\delta^+(U)) - y(\delta^-(U)) \leq f(U)\ \forall U \in \mathcal{C}$, so the claim follows. $\Diamond$

Let $Q := \{y \in \mathbb{R}^A : y(\delta^+(U)) - y(\delta^-(U)) \leq f(U)\ \forall U \in \mathcal{C}\}$.

Claim 2. *For every integer $k \geq 1$, kQ is box-integral.*

Proof of Claim. It suffices to prove this for $k=1$, given that kf is a crossing submodular function for every integer $k \geq 1$. Take $\ell, u \in \mathbb{Z}^A$ such that $\ell \leq u$ and consider a nonempty face of $Q \cap [\ell, u]$, say $F = Q \cap \{y \in \mathbb{R}^A : y(\delta^+(U)) - y(\delta^-(U)) = f(U)\ \forall U \in \mathcal{D},\ y_e = \ell_e\ \forall e \in A_\ell,\ y_e = u_e\ \forall e \in A_u\}$, where $\mathcal{D} \subseteq \mathcal{C}$, $A_\ell, A_u \subseteq A$, $A_\ell \cap A_u = \emptyset$. We need to show that F contains an integer point. To this end, let $\mathcal{C}_1 := \mathcal{C}$ and for every $U \in \mathcal{C}_1$ define

$$f_1(U) := f(U) + \ell(\delta^-(U) \cap A_\ell) - \ell(\delta^+(U) \cap A_\ell) + u(\delta^-(U) \cap A_u) - u(\delta^+(U) \cap A_u).$$

Let $A' := A - (A_\ell \cup A_u)$, $D' := (V, A')$, $\mathcal{C}_2 := \{U \subsetneq V : U \neq \emptyset\}$, and $f_2(U) := u(\delta^+_{D'}(U)) - \ell(\delta^-_{D'}(U))$ for all $U \in \mathcal{C}_2$. Observe that f_1 is a crossing submodular function, because it is the sum of a crossing submodular function f, and crossing modular functions $\ell(\delta^-(U) \cap A_\ell) - \ell(\delta^+(U) \cap A_\ell)$ and $u(\delta^-(U) \cap A_u) - u(\delta^+(U) \cap A_u)$. Observe also that $f_2(U)$, as the sum of a crossing modular function $u(\delta^+_{D'}(U)) - u(\delta^-_{D'}(U))$ and a crossing submodular function $(u-\ell)(\delta^-_{D'}(U))$, is a crossing submodular function.

Consider the polyhedron $P' := \{y \in \mathbb{R}^{A'} : y(\delta^+_{D'}(U)) - y(\delta^-_{D'}(U)) \leq f_i(U)\ \forall U \in \mathcal{C}_i,\ i=1,2\}$ and its face $F' := P' \cap \{y \in \mathbb{R}^{A'} : y(\delta^+_{D'}(U)) - y(\delta^-_{D'}(U)) = f_1(U)\ \forall U \in \mathcal{D}\}$. Note that F' is nonempty since it contains the restriction to $\mathbb{R}^{A'}$ of any point in F. We shall apply Theorem 3 (b) to P'. Clearly the cut condition (2) is satisfied, which in particular implies that $\min_{i=1,2} f_i(U) \leq 0$ for all $U \subsetneq V, U \neq \emptyset$ such that $\delta^+_{D'}(U) = \delta^-_{D'}(U) = \emptyset$. Hence, by Theorem 3 (b), F' contains an integer point $y^\star \in \mathbb{R}^{A'}$ such that $\ell_e \leq y^\star_e \leq u_e$ for all $e \in A'$. Extend the point $y^\star$ to $\mathbb{R}^A$ by defining $y^\star_e := \ell_e$ for all $e \in A_\ell$, and $y^\star_e := u_e$ for all $e \in A_u$. Then $y^\star \in F \cap \mathbb{Z}^A$, as desired. $\Diamond$

It follows from Claim 2 that Q is principally box-integral. This, together with Claim 1, implies that $y(\delta^+(U)) - y(\delta^-(U)) \leq f(U)\ \forall U \in \mathcal{C}$ is box-TDI. $\square$

The second application is the (new) implication that Theorem 3 (a) has for weakly connected digraphs.

Theorem 7. *Let $D = (V, A)$ be a weakly connected digraph and, for $i = 1, 2$, let $\mathcal{C}_i$ be a crossing family over ground set V and $f_i : \mathcal{C}_i \to \mathbb{R}$ be a crossing submodular function. Then the system in* (1) *is TDI, and in particular, the polyhedron P is integral.* □

The third application is the main result of the paper, which is presented in the next section.

3 Proof of main result, and applications

We are ready to prove the main result of our paper, Theorem 2, which we restate for convenience.

> *Let $\tau, k \geq 1$ be integers. Let $D = (V, A)$ be a digraph where $|\delta^+(U)| + (\frac{\tau}{k} - 1)|\delta^-(U)| \geq \tau$ for all $U \subsetneq V, U \neq \emptyset$. Let $\mathcal{C}$ be a crossing family over ground set V, and let $f : \mathcal{C} \to \mathbb{Z}$ be a crossing submodular function such that $f(U) \geq \frac{k}{\tau}(|\delta^+(U)| - |\delta^-(U)|)$ for all $U \in \mathcal{C}$. Then D has a k-arc-connected flip J such that $f(U) \geq |J \cap \delta^+(U)| - |J \cap \delta^-(U)|$ for all $U \in \mathcal{C}$.*

Proof of Theorem 2. Let $\mathcal{C}_1 := \mathcal{C}$, $f_1 := f$, and define $\mathcal{C}_2 := \{U \subsetneq V : U \neq \emptyset\}$, $f_2(U) := |\delta^+(U)| - k$ for all $U \in \mathcal{C}_2$. Observe that f_2 is a crossing submodular function. Consider the vector $y \in \mathbb{R}^A$ that assigns $\frac{k}{\tau}$ to every arc $a \in A$. Then $y(\delta^+(U)) - y(\delta^-(U)) \leq f_1(U)$ for all $U \in \mathcal{C}_1$, by one of our assumptions. Moreover, for all $U \subsetneq V$, $U \neq \emptyset$, our assumption implies that $|\delta^+(V \setminus U)| + (\frac{\tau}{k} - 1)|\delta^-(V \setminus U)| \geq \tau$, which in turn can be written as $y(\delta^+(V \setminus U)) - y(\delta^-(V \setminus U)) \geq k - |\delta^-(V \setminus U)|$, which is equivalent to $y(\delta^+(U)) - y(\delta^-(U)) \leq f_2(U)$. Furthermore, $f_2(U) \leq |\delta^+(U)|$ for all $U \subsetneq V$, $U \neq \emptyset$. It therefore follows from Theorem 3 (b) with $\ell = \mathbf{0}$ and $u = \mathbf{1}$ that there exists $y^\star \in \{0, 1\}^A$ such that $y^\star(\delta^+(U)) - y^\star(\delta^-(U)) \leq f_i(U)$ for all $U \in \mathcal{C}_i$, $i = 1, 2$. Let $J := \{a \in A : y^\star_a = 1\}$. Then $|J \cap \delta^+(U)| - |J \cap \delta^-(U)| = y^\star(\delta^+(U)) - y^\star(\delta^-(U)) \leq f(U)$ for all $U \in \mathcal{C}$. Moreover, $|J \cap \delta^+(U)| - |J \cap \delta^-(U)| \leq f_2(U) = |\delta^+(U)| - k$ for all $U \subsetneq V, U \neq \emptyset$, implying in turn that J is a k-arc-connected flip. Thus, J is the desired set. □

We now present several applications of the main theorem to graph orientations and combinatorial optimization.

First application. For $\tau = 2k$, Theorem 2 gives the following strengthening of the weak orientation theorem.

Theorem 8. *Let $D = (V, A)$ be a digraph whose underlying graph is $2k$-edge-connected. Let $\mathcal{C}$ be a crossing family over ground set V, and let $f : \mathcal{C} \to \mathbb{Z}$ be a crossing submodular function such that $f(U) \geq \frac{1}{2}(|\delta^+(U)| - |\delta^-(U)|)$ for all $U \in \mathcal{C}$. Then D has a k-arc-connected flip J such that $f(U) \geq |J \cap \delta^+(U)| - |J \cap \delta^-(U)|$ for all $U \in \mathcal{C}$.* □

A digraph is *near-Eulerian* if at every vertex the in-degree and out-degree differ by at most one. Theorem 8 implies the following well-known extension of the weak orientation theorem.

Theorem 9 ([14])**.** *Let $D = (V, A)$ be a digraph whose underlying graph is $2k$-edge-connected. Then there exists a k-arc-connected flip J such that after flipping its arcs the digraph becomes near-Eulerian.*

Proof. Let $\mathcal{C} := \{\{u\} : u \in V\} \cup \{V - \{u\} : u \in V\}$, and $f(U) := \left\lceil \frac{1}{2}(|\delta^+(U)| - |\delta^-(U)|) \right\rceil$ for all $U \in \mathcal{C}$. Clearly, $\mathcal{C}$ is a crossing family, f is a crossing submodular function, and $f(U) \geq \frac{1}{2}(|\delta^+(U)| - |\delta^-(U)|)$ for all $U \in \mathcal{C}$. It therefore follows from Theorem 8 that there exists a k-arc-connected flip J such that $f(U) \geq |J \cap \delta^+(U)| - |J \cap \delta^-(U)|$ for all $U \in \mathcal{C}$. In other words, for every vertex $u \in V$,

$$\left\lfloor \frac{1}{2}\left(|\delta^+(u)| - |\delta^-(u)|\right) \right\rfloor \leq |J \cap \delta^+(u)| - |J \cap \delta^-(u)| \leq \left\lceil \frac{1}{2}\left(|\delta^+(u)| - |\delta^-(u)|\right) \right\rceil;$$

that is, the digraph obtained after flipping the arcs in J is near-Eulerian, as required. □

In fact, even in the conclusion of Theorem 8 one can guarantee that after flipping the arcs in J the digraph becomes near-Eulerian. This is obtained by updating $\mathcal{C} := \mathcal{C} \cup \{\{u\} : u \in V\} \cup \{V - \{u\} : u \in V\}$ and $f(U) := \left\lceil \frac{1}{2}(|\delta^+(U)| - |\delta^-(U)|) \right\rceil$ for all $U \in \{\{u\} : u \in V\} \cup \{V - \{u\} : u \in V\}$, and then applying Theorem 8 to the updated crossing family and crossing submodular function.

Dicuts, dijoins, and k-dijoins. Before discussing the second application we need to set up the scene. Let $D = (V, A)$ be a digraph. A *dicut* is an arc subset of the form $\delta^+(U)$ where $\delta^-(U) = \emptyset$, for some $U \subsetneq V, U \neq \emptyset$. A *dijoin* is a subset $J \subseteq A$ that intersects every dicut at least once. Equivalently, J is a dijoin if bidirecting every arc in J makes the digraph D strongly connected. In contrast, J is a 1-arc-connected flip if flipping every arc in J makes the digraph strongly connected. Thus, every 1-arc-connected flip is also a dijoin. It can be readily checked that the converse is not necessarily true. Interestingly, however, any inclusionwise minimal dijoin is a 1-arc-connected flip (see [17], Theorem 55.1). For an integer $k \geq 1$, a *k-dijoin* is an arc subset that intersects every dicut at least k times. Observe that the union of any k disjoint dijoins is a k-dijoin (the converse is not necessarily true, see Figure 1). Moreover, we have the following important observation.

Remark 10. *Given a digraph and an integer $k \geq 1$, every k-arc-connected flip is a k-dijoin.*

(The converse of this remark is not necessarily true even if the k-dijoin is inclusionwise minimal, see Figure 1).

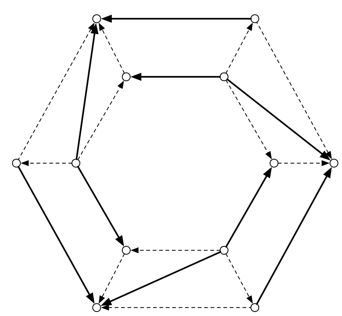

Figure 1: The solid arcs form an inclusionwise minimal 2-dijoin that cannot be decomposed into two dijoins [15], nor is it a 2-arc-connected flip.

Woodall's conjecture and a weaker variant. A seminal result of Lucchesi and Younger is that the minimum size of a dijoin is equal to the maximum number of pairwise disjoint dicuts [11]. Woodall conjectured that the dual minimax relation also holds, that the minimum size of a dicut is equal to the maximum number of pairwise disjoint dijoins, though this remains open [19]. As a step towards Woodall's conjecture, it was recently shown that if the minimum size of a dicut is τ, then A may be decomposed into a dijoin and a $(\tau-1)$-dijoin [1]. In fact, if Woodall's conjecture is true, then one should be able to decompose A into a k-dijoin and a $(\tau-k)$-dijoin, for any integer $k \in \{1,\ldots,\tau-1\}$, but surprisingly even this remains open for $k \neq 1, \tau-1$. This leads us to the following weaker conjecture.

Conjecture 11. *Let $\tau \geq 2$ be an integer. Let $D=(V,A)$ be a digraph where every dicut has size at least τ. Then A can be decomposed into a k-dijoin and a $(\tau-k)$-dijoin, for any $k \in \{1,\ldots,\tau-1\}$.*

Second application. Theorem 2 has the following consequence that relates to Conjecture 11.

Theorem 12. *Let τ, k be integers such that $\tau - 1 \geq k \geq 1$. Let $D=(V,A)$ be a digraph where $|\delta^+(U)| + (\frac{\tau}{k}-1)|\delta^-(U)| \geq \tau$ for all $U \subsetneq V, U \neq \emptyset$. Then A can be decomposed into a k-arc-connected flip and a $(\tau-k)$-dijoin.*

Proof. Let $\mathcal{C}$ be the family of subsets $U \subseteq V$ such that $\delta^+(U)$ is a dicut. Then $\mathcal{C}$ is a crossing family, and $U \mapsto |\delta^+(U)|$ is a crossing submodular (in fact, modular) function over $\mathcal{C}$. The latter implies that the function $f(U) := |\delta^+(U)| - (\tau-k)$ is also a crossing submodular function. The inequalities $|\delta^+(U)| + (\frac{\tau}{k}-1)|\delta^-(U)| \geq \tau$ for all $U \subsetneq V, U \neq \emptyset$, imply that $|\delta^+(U)| \geq \tau$ for all $U \in \mathcal{C}$, which in turn imply that $f(U) \geq \frac{k}{\tau}|\delta^+(U)| = \frac{k}{\tau}(|\delta^+(U)| - |\delta^-(U)|)$ for all $U \in \mathcal{C}$. Thus, we may apply Theorem 2 to get a k-arc-connected flip J such that $|J \cap \delta^+(U)| \leq |\delta^+(U)| - (\tau-k)$ for all $U \in \mathcal{C}$, implying in turn that $A - J$ is a $(\tau-k)$-dijoin, thereby proving Theorem 12. □

Observe that the complement of a k-arc-connected flip is also a k-arc-connected flip. Thus, for $\tau = 2k$, Theorem 12 reduces to the weak orientation theorem, so we get an extension of this theorem in a different direction than Theorem 8.

Special cases of Theorem 12. The inequalities on the cuts of D in Theorem 12 are imposed for the simple reason that they are needed for the proof. That said, they possess some nice properties. For instance, the inequalities ask that the vector $y = \frac{k}{\tau}\mathbf{1}$ satisfies $y(\delta^+(U)) - y(\delta^-(U)) \geq k - |\delta^-(U)|$ for all $U \subsetneq V$, $U \neq \emptyset$. As k increases, the inequalities become more strict: for $k=1$ the inequalities ask precisely that every dicut has size at least τ, so we obtain Theorem 13 below, while for $k = \lfloor \tau/2 \rfloor$ the inequalities *almost* ask that every cut has size at least τ, in turn yielding Theorem 14 also below.

Theorem 13 ([1]). *Let $\tau \geq 2$ be an integer. Let $D = (V, A)$ be a digraph where every dicut has size at least τ. Then A can be decomposed into a dijoin J and a $(\tau - 1)$-dijoin J'.*

This theorem was proved recently in an attempt to prove Woodall's conjecture by first reducing the problem to a special class of *sink-regular* $(\tau, \tau + 1)$*-bipartite digraphs* [1]. The proof we have given here bypasses this reduction.

The reader may wonder why this theorem does not automatically prove Woodall's conjecture, as one may try to repeat the argument on the subdigraph $D \setminus J$. A key complication comes from the fact that deleting an arc from D may create a new dicut, whose size may unfavourably be smaller than $\tau - 1$. Another comes from the fact that given the decomposition $J \cup J'$, one may not necessarily be able to further decompose J' into $\tau - 1$ dijoins [1].

Theorem 14. *Let $\tau \geq 2$ be an integer. Let $D = (V, A)$ be a digraph where every dicut has size at least τ. Suppose further that every cut of D has size at least $\tau - 1$, and if equality holds, then the number of outgoing arcs is equal to the number of incoming arcs. Then A can be decomposed into a k-arc-connected flip and a $(\tau - k)$-dijoin, for any $k \in \{1, \ldots, \lfloor \tau/2 \rfloor\}$.* □

In particular, this proves Conjecture 11 when the underlying graph is τ-edge-connected.

Theorem 15. *Let $\tau \geq 2$ be an integer. If $D = (V, A)$ is a digraph whose underlying graph is τ-edge-connected, then A can be decomposed into a k-dijoin and a $(\tau - k)$-dijoin, for any $k \in [\tau - 1]$.* □

This suggests that it may be easier to prove Woodall's conjecture for τ-edge-connected instances. After all, if the underlying graph has τ disjoint spanning trees, which is guaranteed by 2τ-edge-connectivity for instance [13, 18], then the digraph has τ disjoint dijoins. It should be noted that Woodall's conjecture for $\tau = 3$ has been proven for 4-edge-connected instances, that is, if the underlying graph is 4-edge-connected, then the digraph contains 3 disjoint dijoins [12].

Third application. Finally, Theorem 2 leads to an intriguing extension of Theorem 12 to a setting where arcs are assigned nonnegative integer weights, viewed as capacities for packing dijoins. By replacing an arc of weight $t \geq 1$ with t parallel arcs of weight 1, we may reduce to $0, 1$ weights, so we can focus solely on them. Given a digraph $D = (V, A)$ and $J \subseteq A$, denote by $D[J]$ the subdigraph with vertex set V and arc set J.

Theorem 16. *Let τ, k be integers such that $\tau - 1 \geq k \geq 1$. Let $D = (V, A)$ be a digraph, and $w \in \{0,1\}^A$. Suppose $w(\delta^+(U)) + (\frac{\tau}{k} - 1)w(\delta^-(U)) \geq \tau$ for all $U \subsetneq V$, $U \neq \emptyset$. Then $\{a \in A : w_a = 1\}$ can be decomposed into a k-arc-connected flip of $D[\{a \in A : w_a = 1\}]$ and a $(\tau - k)$-dijoin of D.*

Proof. The proof is similar to that of Theorem 12. This time, however, we apply Theorem 2 to the digraph $D[\{a \in A : w_a = 1\}]$, the crossing family $\{U \subsetneq V : \delta^-_D(U) = \emptyset, U \neq \emptyset\}$, and the crossing submodular function $U \mapsto w(\delta^+_D(U)) - (\tau - k)$. □

Special case of Theorem 16. By specializing Theorem 16 to $k = 1$ we obtain the following.

Theorem 17. *Let $D = (V, A)$ be a digraph, and $w \in \{0,1\}^A$. Suppose $\min\{w(\delta^+(U)), w(\delta^-(U))\} \geq 1$ or $\max\{w(\delta^+(U)), w(\delta^-(U))\} \geq \tau$ for all $U \subsetneq V$, $U \neq \emptyset$. Then $\{a \in A : w_a = 1\}$ can be decomposed into a dijoin and a $(\tau - 1)$-dijoin of D.* □

This theorem connects intriguingly to a conjecture of Chudnovsky, Edwards, Kim, Scott, and Seymour [4] for $0,1$-weighted digraphs (D, w), that if every cut has nonzero weight and every dicut has weight at least τ, then $\{a \in A : w_a = 1\}$ can be decomposed into τ dijoins of D. In fact, another result of ours, namely Theorem 7, has a direct application to this conjecture for $\tau = 2$. In this case, the conjecture can be equivalently formulated as follows; we need a new notion. A *lattice family* $\mathcal{C}$ over a finite ground set V is one where for all $U, W \in \mathcal{C}$ we have $U \cap W, U \cup W \in \mathcal{C}$. Observe that $\mathcal{C} \setminus \{\emptyset, V\}$ is a crossing family.

Conjecture 18 ([4]). *Let $G = (V, E)$ be a spanning tree, and let $\mathcal{C}$ be a lattice family over ground set V such that $|\delta(U)| \geq 2$ for all $U \in \mathcal{C} \setminus \{\emptyset, V\}$. Then there exists an orientation D of G such that $\delta^+_D(U), \delta^-_D(U) \neq \emptyset$ for all $U \in \mathcal{C} \setminus \{\emptyset, V\}$.*

Let D be an arbitrary orientation of G. Consider the system

$$
\begin{aligned}
y(\delta^+_D(U)) - y(\delta^-_D(U)) &\leq |\delta^+_D(U)| - 1 \quad \forall U \in \mathcal{C} \setminus \{\emptyset, V\} \\
y(\delta^+_D(U)) - y(\delta^-_D(U)) &\geq 1 - |\delta^-_D(U)| \quad \forall U \in \mathcal{C} \setminus \{\emptyset, V\}.
\end{aligned}
$$

Observe that $y = \frac{1}{2} \cdot \mathbf{1}$ is a feasible solution. Since D is weakly connected, it follows from Theorem 7 that the system is TDI, and so it has an integral solution. Conjecture 18 states equivalently that the system has a $0, 1$ solution!

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A Two submodular flow systems: TDI-ness and nonnegativity

In general, given a digraph $D = (V, A)$ and two crossing submodular functions $f_i : \mathcal{C}_i \to \mathbb{Z}, i = 1, 2$, consider the combined system $y(\delta^+(U)) - y(\delta^-(U)) \leq f_i(U)\ \forall U \in \mathcal{C}_i, i = 1, 2$.

The combined system is not necessarily integral. To see this, consider the digraph D with vertices $1, 2, 3, 1', 2', 3'$ and arcs $a := 11', b := 22', c := 33'$. Let $\mathcal{C}_1 := \{\{1,2\}, \{1,2,3,1'\}\}$ and $\mathcal{C}_2 := \{\{1,2\}, \{1,2,3,2'\}\}$, which are clearly crossing families. Let $f_1(\{1,2\}) = f_1(\{1,2,3,1'\}) = 1$ and $f_2(\{1,2\}) = f_2(\{1,2,3,2'\}) = 1$, which yield integer-valued crossing submodular functions over

$\mathcal{C}_1, \mathcal{C}_2$, respectively. Then the combined system is $y_a + y_b \leq 1, y_b + y_c \leq 1, y_c + y_a \leq 1$, which is not integral (and therefore not box-TDI) as $(0.5, 0.5, 0.5)$ is a vertex of the polyhedron.

Finding a nonnegative integral solution to the combined system is NP-hard. Take four matroids over the same ground set V with rank functions r_1, r_2, r_3, r_4, respectively, where the functions are given via an oracle (which for every $X \subseteq V$ outputs $r_i(X)$ in unit time). Suppose further $r_i(V) = r$ for all i. It is known that finding a common basis of the four matroids is NP-hard. For example, it includes the NP-complete problems *Does a bipartite graph have a Hamilton cycle?* [2] and *Does a digraph have a Hamilton st-dipath?* (see [9], §3.1.3). (In fact, these two problems prove that finding a common basis of three matroids is NP-hard.)

Let $V^\star$ be a copy of V. Let D be the digraph over vertex set $V \cup V^\star$, and arc set $A := \{(u, u^\star) : u \in V\}$. Let $\mathcal{C} := \{U : U \subseteq V\} \cup \{V \cup \overline{U^\star} : U \subseteq V\} \cup \{V^\star\}$, where $\overline{U^\star} = V^\star - U^\star$. It can be readily checked that $\mathcal{C}$ is a crossing family.

Define $f : \mathcal{C} \to \mathbb{Z}$ as follows: $f(U) := r_1(U)$ for all $U \subseteq V$, $f(V \cup \overline{U^\star}) := r_2(U)$ for all $U \subseteq V$, and $f(V^\star) := -r_1(V)$ (note $f(V) = r$ and $f(V^\star) = -r$). It can be readily checked that f is crossing submodular. Suppose $x \in \mathbb{Z}_+^A$ is an f-submodular flow in D. It can be readily checked that $y \in \{0,1\}^A$, and $\{u \in V : y_{(u,u^\star)} = 1\}$ is a common basis of M_1, M_2. Conversely, for any common basis B of M_1, M_2, the incidence vector of $\{(u, u^\star) : u \in B\}$ is an f-submodular flow in $\{0,1\}^A$. Subsequently, there exists an f-submodular flow in $\mathbb{Z}_+^A$ if and only if M_1, M_2 have a common basis.

Similarly, define $g : \mathcal{C} \to \mathbb{Z}$ as follows: $g(U) := r_3(U)$ for all $U \subseteq V$, $f(V \cup \overline{U^\star}) := r_4(U)$ for all $U \subseteq V$, and $g(V^\star) := -r_3(V)$. Then g, too, is a crossing submodular function, and there exists a g-submodular flow in $\mathbb{Z}_+^A$ if and only if M_3, M_4 have a common basis.

Putting it altogether, we get that there exists an $y \in \mathbb{Z}_+^A$ that is both an f- and g-submodular flow if and only if M_1, M_2, M_3, M_4 have a common basis. Since finding a common basis of four matroids is NP-hard, finding a solution to two sets of submodular flow constraints in $\mathbb{Z}_+^A$ is an NP-hard task.

B Theorem 3 and box constraints

Here we demonstrate that the system from Theorem 3, together with box constraints, is not necessarily integral. To this end, consider the digraph $D = (V, A)$ displayed in Figure 2 (left).

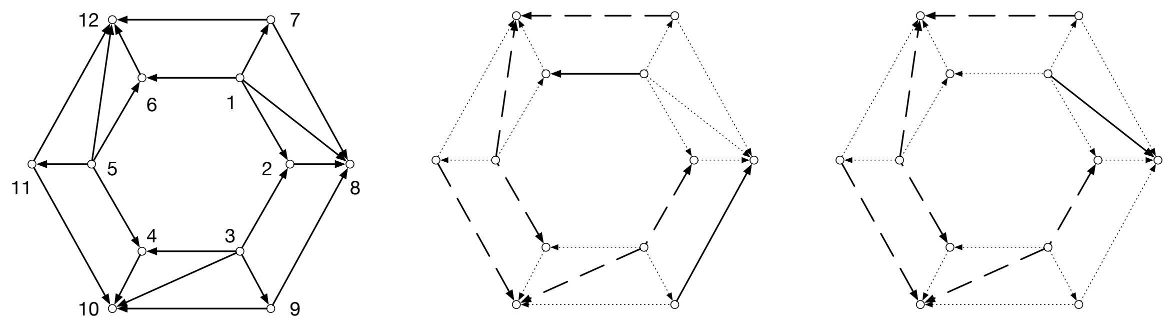

Figure 2: Left: A vertex-labelled digraph $D = (V, A)$. Middle/Right: Representations of two fractional vertices y^1/y^2 of the polytope $Q = P \cap [\mathbf{0}, \mathbf{1}]$, where solid arcs are set to 1, dotted arcs to 0, and dashed arcs to $\frac{1}{2}$.

Define the crossing families $\mathcal{C}_1 := \{U \subsetneq V : \delta^-(U) = \emptyset, U \neq \emptyset\}$ and $\mathcal{C}_2 := \{U \subsetneq V : U \neq \emptyset\}$, and the crossing submodular functions $f_1(U) := |\delta^+(U)| - 3 \; \forall U \in \mathcal{C}_1$ and $f_2(U) := |\delta^+(U)| - 1 \; \forall U \in \mathcal{C}_2$. Clearly $f_2(U) \leq |\delta^+(U)| \; \forall U \in \mathcal{C}_2$. Thus, $\ell = \mathbf{0}$ and $u = \mathbf{1}$ satisfy the cut condition (2). Let P be the polyhedron defined as in (1), and $Q := P \cap [\mathbf{0}, \mathbf{1}]$.

To give some intuition, a $0, 1$ vector $\bar{y}$ belongs to Q if, and only if, $\{a \in A : \bar{y}_a = 0\}$ is a 3-dijoin and $\{a \in A : \bar{y}_a = 1\}$ is a 1-arc-connected flip of D. This gives a description of all the integral vertices of Q. However, Q may have fractional vertices. To see this, define $y^1 \in \{0, \frac{1}{2}, 1\}^A$ where for each $a \in A$, $y^1_a = 0, \frac{1}{2}$ or 1 if a is dotted, dashed, or solid in Figure 2 (middle), respectively. Define $y^2 \in \{0, \frac{1}{2}, 1\}^A$ analogously with respect to Figure 2 (right).

Proposition 19. *y^1, y^2 are vertices of Q.*

Proof. It can be readily checked that y^1, y^2 are feasible. To see that y^i is a vertex, we need to exhibit 21 linearly independent tight inequalities at y^i. We immediately get 15 from $\mathbf{0} \leq y \leq \mathbf{1}$ as y^i has as many coordinates set to 0 or 1. For y^1 the remaining 6 may be chosen as the following inequalities: $y(\delta^+(U)) - y(\delta^-(U)) \leq f_1(U)$ for $U = \{3\}, \overline{\{10\}}, \{5\}, \overline{\{12\}}, \{1, 2, 3, 7, 8, 9\}$

and $y(\delta^+(U)) - y(\delta^-(U)) \le f_2(U)$ for $U = \{7,8,9,10,11,12\}$. For y^2 they may be chosen as $y(\delta^+(U)) - y(\delta^-(U)) \le f_1(U)$ for $U = \{3\}, \overline{\{10\}}, \{5\}, \overline{\{12\}}, \{1,5,6,7,11,12\}$ and $y(\delta^+(U)) - y(\delta^-(U)) \le f_2(U)$ for $U = \{4,5,6,10,11,12\}$. $\square$