The rainbow covering number of clean tangled clutters

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Abstract

In this brief note, we prove a min-min equality for a clean tangled clutter, that the rainbow covering number is equal to the connectivity of its setcore.

Keywords. Clutter, blocker, rainbow covering number, clean clutter.

1 Introduction

A clutter $\mathcal{C}$ over ground set V is *tangled*, or 2-*cover-minimal*, if it has covering number 2, and every element belongs to a minimum cover [ACHL21]. Denote by $G(\mathcal{C})$ the graph over vertex set V whose edges correspond to the minimum covers of $\mathcal{C}$. A cover of $\mathcal{C}$ is *rainbow* if it intersects every connected component of $G(\mathcal{C})$ at most once. The *rainbow covering number of* $\mathcal{C}$, denoted $\mu(\mathcal{C})$, is the minimum size of a rainbow cover of $\mathcal{C}$; if there is no rainbow cover, then $\mu(\mathcal{C}) := +\infty$.

A clutter is *clean* if it has no minor that is a delta or the blocker of an extended odd hole. Let $\mathcal{C}$ be a clean tangled clutter over ground set V. It is known that $G(\mathcal{C})$ is a bipartite graph [AL19]. The *core of* $\mathcal{C}$, denoted $\mathrm{core}(\mathcal{C})$, is the set of all members of $\mathcal{C}$ that intersect every minimum cover exactly once. It is known that the core of $\mathcal{C}$ corresponds uniquely to a set-system in $\{0,1\}^d$, where d is the number of connected components of $G(\mathcal{C})$ – this is defined in §2. The set-system, denoted by $\mathrm{setcore}(\mathcal{C}) \subseteq \{0,1\}^d$, is called a *setcore of* $\mathcal{C}$, and the convex hull of $\mathrm{setcore}(\mathcal{C})$ is a full-dimensional polytope containing $\frac{1}{2}\mathbf{1}$ in its interior [ACS22].

Clean tangled clutters have been a subject of recent study. There is an intriguing interplay between the polyhedral geometry and the combinatorics of clean tangled clutters. For example, the convex hull of the setcore is a simplex if, and only if, the setcore is the cocycle space of a projective geometry over $GF(2)$ [ACS22].

The *connectivity* of a set-system $S \subseteq \{0,1\}^d$, denoted $\lambda(S)$, is the minimum number of variables that appear in a *generalized set covering (GSC)* inequality valid for $\mathrm{conv}(S)$, i.e., an inequality of the form $\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1$ for some $I, J \subseteq [d]$, $I \cap J = \emptyset$; if $S = \{0,1\}^d$, then $\lambda(S) := +\infty$.

In this brief note, we prove that the two parameters that we defined above are equal, further stressing the synergy between the polyhedral geometry and the combinatorics of clean tangled clutters:

Theorem 1.1. *For every clean tangled clutter* $\mathcal{C}$*, we have* $\mu(\mathcal{C}) = \lambda(\mathrm{setcore}(\mathcal{C}))$.

2 Definitions and preliminaries

Clutters Let V be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of V called *members* or *sets*. $\mathcal{C}$ is a *clutter* over *ground set* V if no member contains another [EF70]. A *cover* is a subset $B \subseteq V$ such that $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. The *covering number* of $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. A cover is *minimal* if it does not contain another cover. The *blocker* of $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set V whose members are the minimal covers of $\mathcal{C}$ [EF70]. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [Isb58, EF70]. Take disjoint $I, J \subseteq V$. The *minor* of $\mathcal{C}$ obtained after *deleting* I and *contracting* J, denoted $\mathcal{C} \setminus I/J$, is the clutter over ground set $V-(I\cup J)$ whose members consist of the inclusion-wise minimal sets of $\{C-J : C\in\mathcal{C}, C\cap I=\emptyset\}$. It is well-known that $b(\mathcal{C}\setminus I/J)=b(\mathcal{C})/I\setminus J$ [Sey76].

Clean clutters Two clutters are *isomorphic* if one is obtained from the other by relabeling the ground set. Take an integer $n\geq 3$. Denote by Δ_n the clutter over ground set $[n]:=\{1,\ldots,n\}$ whose members are $\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}$. Any clutter isomorphic to Δ_n is called a *delta of dimension* n. A delta is equal to its blocker. Given an odd integer $n\geq 5$, an *extended odd hole of dimension* n is any clutter whose ground set can be relabeled as $[n]$ so that its minimum cardinality members are precisely $\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}$. Recall that a clutter is clean if it has no minor that is a delta or the blocker of an extended odd hole. Testing cleanness of a clutter belongs to P [ACL20].

Core and setcore Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Recall that $G:=G(\mathcal{C})$ is the graph over vertex set V whose edges correspond to the minimum covers of $\mathcal{C}$. Recall that G is a bipartite graph. Let d be the number of connected components of $G(\mathcal{C})$, and for each $i\in[d]$, denote by $\{U_i,V_i\}$ the bipartition of the i^{th} connected component of G. Recall that the core of $\mathcal{C}$, denoted $\mathrm{core}(\mathcal{C})$, is the set of all members of $\mathcal{C}$ that intersect every minimum cover exactly once.

Theorem 2.1 ([ACS22], Theorem 2.9)**.** $\mathrm{core}(\mathcal{C})=\{C\in\mathcal{C} : C\cap(U_i\cup V_i)\in\{U_i,V_i\}$ *for each* $i\in[d]\}$.

The *setcore of* $\mathcal{C}$ *with respect to* $(U_1,V_1;U_2,V_2;\ldots;U_d,V_d)$ is the set-system $S\subseteq\{0,1\}^d$ that has a point $p\in S$ for every $C\in\mathrm{core}(\mathcal{C})$ such that $p_i=0$ if and only if $C\cap(U_i\cup V_i)=U_i$, for all $i\in[d]$. By Theorem 2.1, the set-system S is well-defined. We denote S by $\mathrm{setcore}(\mathcal{C}:U_1,V_1;U_2,V_2;\ldots;U_d,V_d)$. As the reader can imagine, we will not use this notation often, and use $\mathrm{setcore}(\mathcal{C})$ as short-hand notation. Note however that $\mathrm{setcore}(\mathcal{C})$ is defined only up to isomorphism.

Theorem 2.2 ([ACS22], Theorem 1.5)**.** $\mathrm{conv}(\mathrm{setcore}(\mathcal{C}))$ *is a full-dimensional polytope contained in* $[0,1]^d$ *and containing* $\frac{1}{2}\mathbf{1}$ *in its interior. In particular,* $\mathrm{setcore}(\mathcal{C})$ *does not have duplicated coordinates,* $\mathrm{core}(\mathcal{C})$ *is nonempty, and has covering number two.*

This immediately implies the following.

Corollary 2.3. *If* $d\leq 2$*, then* $\mathrm{setcore}(\mathcal{C})=\{0,1\}^d$. □

We also need the following earlier result.

Theorem 2.4 ([ACGT22], Theorem 2.5, and [ACS22], Lemma 2.6)**.** *Suppose G is not a connected graph. Let $\{U,U'\}$ be the bipartition of a connected component of G. Then $\mathcal{C}\setminus U/U'$ is a clean tangled clutter such that* $\mathrm{core}(\mathcal{C}\setminus U/U')\subseteq\mathrm{core}(\mathcal{C})\setminus U/U'$.

3 Proof of Theorem 1.1

Let $\mathcal{C}$ be a clean tangled clutter over ground set V, let $G:=G(\mathcal{C})$, and let d be the number of connected components of G. For each $i\in[d]$, let $\{U_i,V_i\}$ be the bipartition of the i^{th} connected component of G. Let $\mu:=\mu(\mathcal{C})$ be the rainbow covering number, and let $\lambda:=\lambda(\mathrm{setcore}(C))$ be the connectivity of $\mathrm{setcore}(C)$. We need a key notion.

Definition 3.1. *A* monochromatic cover *of* $\mathcal{C}$ *is a cover that is monochromatic in some proper* 2*-vertex-coloring of* G*. A monochromatic cover of* $\mathcal{C}$*, say of the form* $\bigcup_{i\in I}V_i$ *for some* $I\subseteq[d]$*, is* irreducible *if for each* $j\in I$*,* $\left(\bigcup_{i\in I,i\neq j}V_i\right)\cup U_j$ *is not a cover.*

Observe that every rainbow cover is also monochromatic. We need the following parameters:

$\mu_1(\mathcal{C})$: the minimum size of a cover of $\text{core}(\mathcal{C})$ that is monochromatic in some proper 2-vertex-coloring of G. It can be readily seen that any such cover that is inclusion-wise minimal intersects every component of G at most once, and $\mu \geq \mu_1(\mathcal{C}) = \lambda$.

$\mu_2(\mathcal{C})$: the minimum number of components of G intersected by a monochromatic cover of $\mathcal{C}$.

$\mu_3(\mathcal{C})$: the minimum number of components of G intersected by an irreducible monochromatic cover of $\mathcal{C}$.

Let $\mu_i := \mu_i(\mathcal{C})$ for $i = 0, 1, 2$. To prove Theorem 1.1, it remains to show that $\mu_1 \geq \mu$. The first lemma below is a minor extension of [[ACS22], Lemma 2.8] and the second lemma is related to [[ACS22], Theorem 5.2].

Lemma 3.2. *Suppose for some $u, v \in V$, every member of $\text{core}(\mathcal{C})$ containing u also contains v. Then u, v belong to the same part of the bipartition of a connected component of G.*

Proof. By Theorem 2.1, it suffices to show that u, v belong to the same connected component of G. Suppose otherwise. In particular, G is not connected. Let $\{U, U'\}$ be the bipartition of the connected component containing u where $u \in U'$. Then $\mathcal{C} \setminus U/U'$ is a clean tangled clutter such that $\text{core}(\mathcal{C} \setminus U/U') \subseteq \text{core}(\mathcal{C}) \setminus U/U'$ by Theorem 2.4. Let w be a neighbor of u in G; so $w \in U$. Then $\{w, u\}$ is a cover of $\mathcal{C}$. As every member of $\text{core}(\mathcal{C})$ containing u also contains v, it follows that $\{w, v\}$ is a cover of $\text{core}(\mathcal{C})$, implying in turn that $\text{core}(\mathcal{C}) \setminus U/U'$ has $\{v\}$ as a cover. However, $\text{core}(\mathcal{C} \setminus U/U') \subseteq \text{core}(\mathcal{C}) \setminus U/U'$, so $\text{core}(\mathcal{C} \setminus U/U')$ has a cover of cardinality 1, a contradiction to Theorem 2.2. □

Lemma 3.3. *If $V_1 \cup \cdots \cup V_k$ is an irreducible monochromatic cover for some integer $k \in [r]$, then there exists a monochromatic minimal cover B such that $B \subseteq \bigcup_{i=1}^k V_i$ and $|B \cap V_i| = 1$ for each $i \in [k]$.*

Proof. Out of all the monochromatic minimal covers of $\mathcal{C}$ contained in $\bigcup_{i=1}^k V_i$, pick one of minimum cardinality, call it B. As $\bigcup_{i=1}^k V_i$ is an irreducible monochromatic cover, it follows that $B \cap V_i \neq \emptyset, i \in [k]$. To finish the proof of the lemma, it suffices to show that $|B \cap V_1| = 1$. Suppose for a contradiction that $|B \cap V_1| \geq 2$. Let $I := B - V_1$, $J := V - (U_1 \cup V_1 \cup I)$, and $\mathcal{C}' := \mathcal{C} \setminus I/J$, a minor over ground set $U_1 \cup V_1$. Assume in the first case that $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is clean and tangled, and $G[U_1 \cup V_1] \subseteq G(\mathcal{C}')$. Thus $G(\mathcal{C}')$ is a connected bipartite graph whose bipartition is inevitably $\{U_1, V_1\}$. It therefore follows from Corollary 2.3 that $U_1, V_1 \in \mathcal{C}'$. However, $B \cap V_1 = B - I$ is a cover of $\mathcal{C}'$ disjoint from U_1, a contradiction. Assume in the remaining case that $\tau(\mathcal{C}') \leq 1$. That is, there is a $D \in b(\mathcal{C})$ such that $D \subseteq U_1 \cup V_1 \cup I$ and $|D - I| \leq 1$. As $D \subseteq (V_1 \cup \cdots \cup V_k) \cup U_1$, and $V_1 \cup \cdots \cup V_k$ is an irreducible monochromatic cover, it follows that $D \subseteq \bigcup_{i=1}^k V_i$. But then D is a monochromatic minimal cover of $\mathcal{C}$ contained in $\bigcup_{i=1}^k V_i$ and

$$|D| = |D - I| + |D \cap I| \leq 1 + |B - (U_1 \cup V_1)| < |B \cap (U_1 \cup V_1)| + |B - (U_1 \cup V_1)| = |B|,$$

a contradiction to our minimal choice of B. As a result, $|B \cap V_1| = 1$, as desired. □

Lemma 3.4. *The following inequalities hold:*

1. $\mu_1 \geq \mu_2$,
2. $\mu_2 \geq \mu_3$,
3. $\mu_3 \geq \mu$.

Proof. **(1)** If $\mu_1 = \infty$, then the inequality $\mu_2 \leq \mu_1$ holds clearly. Otherwise, μ_1 is finite.

We claim that $\mu_1 \geq 3$. For if not, then $\text{core}(\mathcal{C})$ would have a cover $\{u, v\}$ of size 2 that is monochromatic in some proper 2-vertex-coloring of G. Clearly, u, v must be from different connected components of G. Let v' be an element in V such that $\{v, v'\}$ is an edge of G. Then every member of $\text{core}(\mathcal{C})$ containing v' does not contain v so it must contain u. Subsequently, by Lemma 3.2, u and v', and therefore u and v, are from the same connected component of G, a contradiction.

We prove by induction on μ_1 that $\mu_2 \leq \mu_1$. If $\mu_2 \leq 3$, then the inequality follows from the inequality $\mu_1 \geq 3$ that we just showed. For the induction step, we assume that $\mu_2 > 3$. Suppose $U_1 \cup \cdots \cup U_{\mu_1}$ is a cover of $\mathrm{core}(\mathcal{C})$, and let $\mathcal{C}' := \mathcal{C} \setminus U_{\mu_1}/V_{\mu_1}$, which is also a clean tangled clutter. As $\mu_2 > 3$, it follows that $G(\mathcal{C}')$ has the same connected components as G except for $U_{\mu_1} \cup V_{\mu_1}$.

By the induction hypothesis, $\mu_2(\mathcal{C}') \leq \mu_1(\mathcal{C}')$. On the one hand, $\mu_1(\mathcal{C}') \leq \mu_1 - 1$ as $U_1 \cup \cdots \cup U_{\mu_1 - 1}$ is a cover of $\mathrm{core}(\mathcal{C}')$; this is because $\mathrm{core}(\mathcal{C}') \subseteq \mathrm{core}(\mathcal{C}) \setminus U_{\mu_1}/V_{\mu_1}$ by Theorem 2.4 and $U_1 \cup \cdots \cup U_{\mu_1 - 1}$ is clearly a cover of the latter. On the other hand, $\mu_2(\mathcal{C}') \geq \mu_2 - 1$ as any monochromatic cover U of $\mathcal{C}'$ yields a monochromatic cover of $\mathcal{C}$, namely $U \cup U_{\mu_1}$, intersecting only 1 more component of G. Thus, $\mu_1 - 1 \geq \mu_1(\mathcal{C}') \geq \mu_2(\mathcal{C}') \geq \mu_2 - 1$, implying in turn that $\mu_1 \geq \mu_2$, thereby completing the induction step.

(2) If $\mu_2 = \infty$, then we are done. Otherwise, suppose $U_1 \cup \cdots \cup U_{\mu_2}$ is a monochromatic cover of $\mathcal{C}$. Suppose for a contradiction that $U_1 \cup \cdots \cup U_{\mu_2 - 1} \cup U_{\mu_2}$ is not irreducible, say $U_1 \cup \cdots \cup U_{\mu_2 - 1} \cup V_{\mu_2}$ is also a cover of $\mathcal{C}$. Then $U_1 \cup \cdots \cup U_{\mu_2 - 1}$ must be a cover of $\mathrm{core}(\mathcal{C})$, implying in turn that $\mu_2 - 1 \geq \mu_1$, thus contradicting (1).

(3) If $\mu_3 = \infty$, we are done. Otherwise, the inequality follows from Lemma 3.3. □

We are ready to prove the promised relation, that $\mu = \lambda$.

Proof of Theorem 1.1. By Lemma 3.4, $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu$. We also know that $\mu \geq \mu_1 = \lambda$, thus $\mu = \lambda$, as required. □

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