On dyadic fractional packings of T-joins

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July 25, 2022

Abstract

Let $G = (V, E)$ be a graph, and $T \subseteq V$ a nonempty subset of even cardinality. The famous theorem of Edmonds and Johnson on the T-join polyhedron implies that the minimum cardinality of a T-cut is equal to the maximum value of a fractional packing of T-joins. In this paper, we prove that the fractions assigned may be picked as *dyadic* rationals, i.e. of the form $\frac{a}{2^k}$ for some integers $a, k \geq 0$.

1 Introduction

Let $G = (V, E)$ be a graph, and $T \subseteq V$ a nonempty subset of even cardinality. The vertices in T will be referred to as *terminals*. For $J \subseteq E$, denote by $\mathrm{odd}(J)$ the set of odd degree vertices of $G[J]$, the subgraph of G with edge set J. J is said to be a *T-join* if $\mathrm{odd}(J) = T$. Given edge weights $w \in \mathbb{Z}^E$, the *weight of a T-join* is the sum of the weights of its edges. What is the minimum weight of a T-join?

This question was asked due to its connections with some important problems in Combinatorial Optimization, namely the *minimum weight perfect matching problem*, equivalent to the *maximum weight matching problem* [7], and the *Chinese postman set problem*, where one looks for a minimum cardinality closed walk through all the edges of a connected graph [10, 8].

The T-join problem can be modeled by an elegant set covering linear program. A *T-cut* is a cut of the form $\delta(U) \subseteq E$, where $U \subseteq V$ and $|U \cap T|$ is odd. It can be readily checked that every T-join and T-cut have an odd, and therefore nonzero, number of edges in common. It therefore follows that the set covering linear program below (left) has the incidence vector of every T-join as a feasible solution:

$$\begin{array}{lll} \min & w^\top x & \\ \text{s.t.} & \sum(x_e : e \in B) \geq 1 & \forall B \text{ a } T\text{-cut} \\ & x_e \geq 0 & \forall e \in E \end{array} \qquad \begin{array}{lll} \max & \mathbf{1}^\top y & \\ \text{s.t.} & \sum(y_B : e \in B) \leq w_e & \forall e \in E \\ & y_B \geq 0 & \forall B \text{ a } T\text{-cut.} \end{array}$$

The famous theorem of Edmonds and Johnson states that this linear program (left) exactly models the minimum weight T-join problem, that it has an integral optimal solution corresponding to the incidence vector of a T-join [8].

What can we say about the dual linear program above (right), does it always have an integral optimal solution? The answer turns out to be No. This can be seen by looking at the complete graph K_4, where every vertex is a terminal and every edge weight is 1; then the dual has a unique optimal solution, one that assigns $\frac{1}{2}$ to every T-cut. Fortunately, this is as bad as it gets; in general, the dual linear program always has a half-integral optimal solution [12, 15].

Let us now swap the roles of T-joins and T-cuts, and consider the following primal-dual pair instead:

$$(P)\ \begin{array}{lll} \min & w^\top x & \\ \text{s.t.} & \sum(x_e : e \in J) \geq 1 & \forall J \text{ a } T\text{-join} \\ & x_e \geq 0 & \forall e \in E \end{array} \qquad (D)\ \begin{array}{lll} \max & \mathbf{1}^\top y & \\ \text{s.t.} & \sum(y_J : e \in J) \leq w_e & \forall e \in E \\ & y_J \geq 0 & \forall J \text{ a } T\text{-join.} \end{array}$$

The incidence vector of every T-cut is a feasible solution to (P). In fact, the *theory of blocking polyhedra* tells us that (P) exactly models the *minimum weight T-cut problem*, that is, (P) has an integral optimal solution corresponding to the incidence vector of a T-cut [11, 9].

Similar to above, does (D) always have an integral or half-integral optimal solution? Unfortunately, unlike for (P), blocking theory fails to give us any information about (D). Things get worse as the answer to the question is No! Seymour constructed an example from the Petersen graph for which the dual above has no integral or half-integral (or even $\frac{1}{3}$-integral) optimal solution, but it does have quarter-integral optimal solution [16]. Recently, it was shown that if the optimal value is equal to two, then (D) has a quarter-integral optimal solution [3, 4]. Seymour's example shows that this result is indeed best possible.

One must then revise the question, and may ask the following: Does (D) always have a quarter-integral optimal solution? This would then become a special case of the $\frac{1}{4}$-*MFMC Conjecture* for ideal clutters [6], and would follow from the *generalized Fulkerson conjecture* [16] (see also [5]).

In this paper, however, we address the general number-theoretic behaviour of optimal solutions to (D). A *dyadic rational* is an integer multiple of $\frac{1}{2^k}$ for some integer $k \geq 0$. A vector is *dyadic* if each of its entries is a dyadic rational. In this paper, we prove the following result:

Theorem 1.1. *Let $G = (V, E)$ be a graph, and $T \subseteq V$ a nonempty subset of even cardinality. Then (D) has a dyadic optimal solution.*

This theorem confirms a conjecture of Seymour on ideal clutters, called the *Dyadic Conjecture*, for the clutter of T-joins (see [14], §79.3e, and [1]).

2 Matching lattice and dyadic linear programming

There are two key ingredients for the proof of Theorem 1.1, one being a result of Lovász on the *matching lattice*, another a recent result in *Dyadic Linear Programming*. In this section, we present these ingredients. We then use the ingredients to prove a special case of Theorem 1.1, which will serve as the base case for an inductive proof.

The matching lattice. Take an integer $r \geq 2$. An *r-graph* is an r-regular graph on an even number of vertices such that every *odd cut*, i.e. $\delta(U)$ where $|U|$ is odd, has cardinality at least r. These graphs were introduced by Seymour; he noted that every edge of an r-graph belongs to a perfect matching [16]. Extending an earlier result of Seymour for the $r = 3$ case [16], Lovász proved the following result:

Theorem 2.1 ([13], (6.5))**.** *Let $G = (V, E)$ be an r-graph for some integer $r \geq 2$. Then $2 \cdot \mathbf{1} \in \mathbb{R}^E$ belongs to the* matching lattice *of G. That is, $2 \cdot \mathbf{1}$ can be written as an integer linear combination of the vectors $\{\chi_M : M \subseteq E$ a perfect matching of $G\}$.*

To prove this theorem, Lovász had to utilize the theory of *bricks* and *braces* to its full extent to prove the exceptional role of the Petersen graph as a brick, and only then did he obtain the result above. Now, let $G = (V, E)$ be a graph. A vector $w \in \mathbb{R}^E$ is *matching-integral* for G if $w(M)$ is an integer for every perfect matching M. As an immediate consequence of Theorem 2.1, we get the following result, which is used to prove our main result.

Corollary 2.2. *Let $G = (V, E)$ be an r-graph for some integer $r \geq 2$. If w is matching-integral, then $\mathbf{1}^\top w$ is $\frac{1}{2}$-integral.*[1]

Dyadic linear programming. Another important ingredient comes from a more general question that is relevant in the context of floating-point arithmetic: *Given a linear program, when does it have an optimal solution that is dyadic?* In a recent paper [2], the authors proved (a generalization of) the following theorem, which is used to prove our main result.

Theorem 2.3 ([2], Corollary 16)**.** *Let P be a nonempty rational polyhedron whose affine hull is $\{x : Ax = b\}$, where A, b have integral entries. Then exactly one of the following statements holds:*

1. *P contains a dyadic vector.*
2. *There exists a vector y such that $y^\top A$ is integral and $y^\top b$ is non-dyadic.*[2]

The base case. A cut of $G = (V, E)$ is *trivial* if it is of the form $\delta(v), v \in V$. Let us use the two ingredients just presented to prove the following theorem, which will serve as the base case for an inductive proof of Theorem 1.1.

Theorem 2.4. *Take an integer $r \geq 2$, and let $G = (V, E)$ be an r-graph where every minimum odd cut is trivial. Then there is an assignment of a nonnegative dyadic rational y_M to every perfect matching M such that $\mathbf{1}^\top y = r$, and for each $e \in E$, $\sum(y_M : M \ni e) = 1$.*

Proof. Consider the following primal-dual pair of linear programs:

$$(P)\quad \begin{array}{lll} \min & \mathbf{1}^\top x & \\ \text{s.t.} & \sum(x_e : e \in J) \geq 1 & \forall J \text{ a } V\text{-join} \\ & x_e \geq 0 & \forall e \in E \end{array} \qquad (D)\quad \begin{array}{lll} \max & \mathbf{1}^\top y & \\ \text{s.t.} & \sum(y_J : e \in J) \leq 1 & \forall e \in E \\ & y_J \geq 0 & \forall J \text{ a } V\text{-join.} \end{array}$$

As mentioned in §1, (P) exactly models the minimum V-cut, i.e. odd cut, problem. As G is an r-graph, the optimal value of (P) is r, and by our setup, the extreme optimal solutions of (P) are precisely the incidence vectors of the trivial cuts of G. In particular, (D) has optimal value r.

[1] The sequence in which we presented these two results, though convenient, is misleading. In fact, Lovász proves Corollary 2.2 first, and then obtains Theorem 2.1 as a consequence.

[2] In particular, P contains a dyadic vector if, and only if, its affine hull contains a dyadic vector.

Now let $y \geq \mathbf{0}$. Then by Complementary Slackness, y is an optimal solution to (D) if, and only if, $\sum (y_J : J \ni e) = 1$ for all $e \in E$, and $y_J > 0$ only if J is a perfect matching. In fact, since we have a full description of the optimal solutions of (P), we may use Strict Complementarity to conclude

($\star$) there exists an optimal solution to (D) that is nonzero on every perfect matching.

Now let A be the matrix whose rows are indexed by the edges, and whose columns are the incidence vectors of the perfect matchings of G. By what we just argued, the optimal solutions to (D) are in a one-to-one correspondence with the vectors in the polyhedron $P := \{y : Ay = \mathbf{1}, y \geq \mathbf{0}\}$.

Our objective is to find a dyadic vector in P. To this end, let us apply Theorem 2.3. By ($\star$), P contains a vector that is nonzero on every coordinate, implying in turn that the affine hull of P is $\{y : Ay = \mathbf{1}\}$. Suppose next that $w^\top A$ is integral, that is, w is a matching-integral vector. Then by Corollary 2.2, $w^\top \mathbf{1}$ is $\frac{1}{2}$-integral and so dyadic. Thus, Theorem 2.3 (2) does not hold, so (1) must hold, implying in turn that P contains a dyadic vector, as required. □

3 Proof of the main theorem

In this section, we prove the main theorem. We need a lemma, for which we need a notion. A graph is *Eulerian* if every vertex has even degree.

Lemma 3.1. *Let $G = (V, E)$ be an Eulerian graph, $T \subseteq V$ nonempty and of even cardinality, and τ the minimum cardinality of a T-cut. Suppose $\tau \geq 2$. Let $T' := \{v \in T : \deg(v) = \tau\}$, and suppose every vertex belongs to T' or is adjacent to only vertices in T'. Then there exists a dyadic fractional packing of T-joins of value τ.*

Proof. We proceed by induction on $f(G,T) := |V| + \sum_{v \notin T} \deg(v) + \sum_{v \in T} (\deg(v) - \tau)$. The base case $f(G,T) = 2$ is trivial. For the induction step, assume that $f(G,T) \geq 3$.

Claim 1. *If there exists a minimum T-cut that is not trivial, then there exists a dyadic fractional packing of T-joins of value τ.*

Proof of Claim. Suppose the minimum T-cut $\delta(U)$ is non-trivial, for some $U \subseteq V$ such that $|U \cap T|$ is odd and $|U|, |V - U| \neq 1$. Let G_1, G_2 be the graphs obtained from G after shrinking $U, V - U$ to a single vertex v_1, v_2, respectively. Let $T_1 := \{v_1\} \cup (T - U)$ and $T_2 := \{v_2\} \cup (T \cap U)$. Note that each G_i is an Eulerian graph, every T_i-cut in G_i is also a T-cut in G, and so the minimum cardinality of a T_i-cut is τ. Moreover, for $T'_i := \{v \in T_i : \deg(v) = \tau\}$ which contains v_i, every vertex of G_i is either in T'_i or adjacent to only vertices in T'_i. Observe further that $f(G_i, T_i) \leq f(G, T) - 2$. Thus, by the induction hypothesis, we have an optimal fractional packing $\bar{y}^i$ of T_i-joins in G_i that is dyadic. Since every edge of G_i appears in a minimum T_i-cut, we may apply Complementary Slackness to conclude that for each edge e of G_i,

$$\sum \left(\bar{y}^i_J : J \text{ is a } T_i\text{-join of } G_i \text{ such that } e \in J\right) = 1, \tag{1}$$

and for each T_i-join J of G_i,

$$\text{if } \bar{y}^i_J > 0, \text{ then } |J \cap \delta(U)| = 1. \tag{2}$$

Let us "glue" $\bar{y}^1$ and $\bar{y}^2$ to obtain an optimal fractional packing $\bar{y}$ of T-joins in G that is dyadic.

An *admissible pair* is a pair (J_1, J_2) where each J_i is a T_i-join of G_i, and $J_1 \cap \delta(U) = J_2 \cap \delta(U)$. Observe that for every admissible pair (J_1, J_2), the set $J_1 \cup J_2 \subseteq E$ is a T-join of G; note further that this is a one-to-one mapping, that is, different admissible pairs give rise to different T-joins of G. For each admissible pair (J_1, J_2), let $\bar{y}_{J_1 \cup J_2} := \bar{y}^1_{J_1} \times \bar{y}^2_{J_2}$, which is a nonnegative dyadic rational. Note that $\bar{y}_{J_1 \cup J_2}$ is well-defined as the T-join $J_1 \cup J_2$ is uniquely obtained from the pair (J_1, J_2). For every T-join J of G not considered above, let $\bar{y}_J := 0$. It can be readily checked that as an application of (1) and (2), $\bar{y}$ is an optimal fractional packing of T-joins of G that is dyadic, thereby completing the induction step. $\diamondsuit$

We may therefore assume that every minimum T-cut is trivial. In particular, since G is Eulerian, every non-trivial T-cut has cardinality at least $\tau + 2$.

Claim 2. *If $T' \neq V$, then there exists a dyadic fractional packing of T-joins of value τ.*

Proof of Claim. Let $u \in V - T'$. If u is isolated, then we apply the induction hypothesis to $(G \backslash u, T)$ to get the desired fractional packing. Otherwise, u has at least two neighbours, say x, y. By assumption, $x, y \in T'$. Let H be the graph obtained from G after removing an edge $e = \{u, x\}$ and an edge $f = \{u, y\}$, and adding a new edge $g := \{x, y\}$.

Observe that H is an Eulerian graph. Since every non-trivial T-cut of G has cardinality at least $\tau + 2$, it follows that every minimum T-cut of H has cardinality at least τ. Also, $T' \subseteq \{v \in T : \deg_H(v) = \tau\} \subseteq T' \cup \{u\}$, and every vertex of H either belongs to the set $\{v \in T : \deg_H(v) = \tau\}$ or is adjacent to only the vertices in the set. Since $f(H,T) = f(G,T) - 2$, we may therefore apply the induction hypothesis to conclude that H has a dyadic fractional packing y' of T-joins of value τ. Every T-join J in H that uses g can be turned into the T-join $J \triangle \{e, f, g\}$. This transformation naturally turns y' into an optimal fractional packing y of T-joins in G. Since the entries of y are in correspondence with the entries of y', y is also dyadic, thereby completing the induction step. $\Diamond$

We may therefore assume that $T' = V$. This means that G is a τ-graph where every minimum odd cut is trivial, in which case we get our desired fractional packing from Theorem 2.4, thereby completing the induction step. $\square$

We are now ready to prove Theorem 1.1:

Proof of Theorem 1.1. Let $G = (V, E)$ be a graph, and $T \subseteq V$ nonempty and of even cardinality. Let τ be the minimum cardinality of a T-cut. We need to prove that there exists a dyadic fractional packing of T-joins of value τ. We may assume that $\tau \geq 2$. We proceed by induction on $|V| + |E| \geq 4$. The base case is trivial, so we move on to the induction step.

Claim 1. *If an edge e does not belong to a minimum T-cut, then there exists a dyadic fractional packing of T-joins of value τ.*

Proof of Claim. Observe that the minimum cardinality of a T-cut in $G \setminus e$ remains τ. Thus, by the induction hypothesis, $G \setminus e$, and therefore G, has a dyadic fractional packing of T-joins of value τ. $\Diamond$

Claim 2. *If there exists a minimum T-cut that is not trivial, then there exists a dyadic fractional packing of T-joins of value τ.*

Proof of Claim. Suppose a minimum T-cut $\delta(U)$ is non-trivial, for some $U \subseteq V$ such that $|U \cap T|$ is odd and $|U|, |V - U| \neq 1$. Let G_1, G_2 be the graphs obtained from G after shrinking $U, V - U$ to a single vertex v_1, v_2, respectively. Let $T_1 := \{v_1\} \cup (T - U)$ and $T_2 := \{v_2\} \cup (T \cap U)$. Then the minimum cardinality of a T_i-cut in G_i remains τ. Since $|V(G_i)| + |E(G_i)| \leq |V| + |E| - 1$, the induction hypothesis applies, and implies the existence of an optimal fractional packing y^i of T_i-joins in G_i that is dyadic. By glueing y^1, y^2 along the edges of $\delta(U)$ (similar as in the proof of Lemma 3.1), we get an optimal fractional packing y of T-joins in G that is dyadic, as required. $\Diamond$

We may therefore assume that every edge belongs to a minimum T-cut, and that every such T-cut is trivial. Now double every edge of G to get an Eulerian graph H, where the minimum cardinality of a T-cut is 2τ. Observe that every edge of H also belongs to a minimum T-cut, and that every such T-cut is trivial. In particular, for $T' := \{v \in T : \deg_H(v) = 2\tau\}$, every vertex of H is either in T' or is adjacent to only vertices in T'. Thus, by Lemma 3.1, H has a dyadic fractional packing y' of T-joins of value τ. We may assume the fractional packing assigns nonzero fractions only to minimal T-joins. Since the minimal T-joins use at most one edge of every pair of parallel edges, the fractional packing y' naturally yields an assignment of fractions y to the T-joins of G such that every edge has congestion 2, and $\mathbf{1}^\top y = 2\tau$. Observe now that $\frac{1}{2}y$ is a dyadic fractional packing of T-joins of G of value τ, as desired. This completes the induction step. $\square$

4 A conjecture

Let us conclude with a conjecture. Let $p \geq 3$ be a prime number. A *p-adic rational* is an integer multiple of $\frac{1}{p^k}$, for some integer $k \geq 0$. A natural question is whether there is a p-adic analogue of Theorem 1.1, that is, whether (D) has a p-adic optimal solution? Seymour's example mentioned in §1 answers this question negatively. More precisely, let G be the graph obtained from the Petersen graph P_{10} after subdividing every edge once, and let T be an even cardinality subset of vertices containing all the vertices of degree two. It can be shown that if (D) has a $\frac{1}{k}$-integral optimal solution, then k must be even [16]. In particular, for this example, (D) has no p-adic optimal solution. However, it might be possible that is the only "minimal" counterexample. Let us elaborate.

Let (G, T) be a pair where $G = (V, E)$ is a graph and $T \subseteq V$ a subset of even cardinality. Take an edge $e \in E$. The *deletion* $(G, T) \setminus e$ is the pair $(G \setminus e, T)$, while the *contraction* $(G, T)/e$ is the pair

$(G/e, T')$[3], where $T' = T - e$ if $|e \cap T|$ is even, and $T' = (T - e) \cup \{\text{shrunken vertex}\}$ if $|e \cap T|$ is odd. A *graft minor of* (G, T) is any pair obtained after a sequence of deletions and contractions. We conjecture the following.

Conjecture 4.1. *Let $G = (V, E)$ be a graph, and $T \subseteq V$ a nonempty subset of even cardinality. If (G, T) has no $(P_{10}, V(P_{10}))$ graft minor, then for every prime number $p \geq 3$, there exists an optimal fractional packing of T-joins that is p-adic.*

By using a p-adic analogue of Theorem 2.3, as well as a result of Lovász on matching-covered graphs without a Petersen brick [13], one can prove this conjecture when G is an r-graph, and $T = V$.[4]

Acknowledgements

Gérard Cornuéjols was supported by ONR grant: N00014-22-1-2528. We would like to thank the anonymous referees whose feedback improved the presentation of the paper.

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[3] In this context, to contract a loop is to delete it.

[4] More precisely, for every r-graph without a Petersen brick, there is an assignment of a nonnegative p-adic rational y_M to every perfect matching M such that $\mathbf{1}^\top y = r$, and $\sum (y_M : e \in M) = 1$ for every edge e.

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