Strongly connected orientations and integer lattices

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Abstract

Let $D = (V, A)$ be a digraph whose underlying graph is 2-edge-connected, and let P be the polytope whose vertices are the incidence vectors of arc sets whose reversal makes D strongly connected. We study the lattice theoretic properties of the integer points contained in a proper face F of P not contained in $\{x : x_a = i\}$ for any $a \in A, i \in \{0, 1\}$. We prove under a mild necessary condition that $F \cap \{0, 1\}^A$ contains an *integral basis* B, i.e., B is linearly independent, and any integral vector in the linear hull of F is an integral linear combination of B. This result is surprising as the integer points in F do not necessarily form a Hilbert basis. In proving the result, we develop a theory similar to Matching Theory for degree-constrained dijoins in bipartite digraphs. Our result has consequences for head-disjoint strong orientations in hypergraphs, and also to a famous conjecture by Woodall that the minimum size of a dicut of D, say τ, is equal to the maximum number of disjoint dijoins. We prove a relaxation of this conjecture, by finding for any prime number $p \geq 2$, a p-adic packing of dijoins of value τ and of support size at most $2|A|$. We also prove that the all-ones vector belongs to the lattice generated by $F \cap \{0, 1\}^A$, where F is the face of P satisfying $x(\delta^+(U)) = 1$ for every minimum dicut $\delta^+(U)$.

Keywords: strongly connected orientation, M-convex set, Hilbert basis, integer lattice, integral basis, Woodall's conjecture.

1 Introduction

Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. A *strengthening set* is an arc subset J such that the digraph obtained from D after reversing the arcs in J is strongly connected. Observe that $J \subseteq A$ is a strengthening set if, and only if, its indicator vector x satisfies the following *generalized set covering inequalities*:

$$\sum_{a \in \delta^+(U)} x_a + \sum_{b \in \delta^-(U)} (1 - x_b) \geq 1 \quad \forall U \subset V, U \neq \emptyset. \tag{CUT}$$

In words, (CUT) asks that after reversing the arcs of J in D, every nonempty proper vertex subset U has at least one incoming arc. Observe that (CUT) can be rewritten as $x(\delta^+(U)) - x(\delta^-(U)) \geq 1 - |\delta^-(U)|$; as the right-hand sides correspond to a crossing supermodular function, the system above may be viewed as a *supermodular flow system*. Let

$$\mathrm{SCR}(D) := [0, 1]^A \cap \{x : x \text{ satisfies (CUT)}\}.$$

It is well-known that $\mathrm{SCR}(D)$ is a nonempty integral polytope, and so its vertices are precisely the indicator vectors of the strengthening sets of D ([9], see [22], §60.1). This polytope and its variants have played an important role in graph orientations, combinatorial and matroid optimization; see ([22], Chapters 60-61) and ([11], Chapter 16).

In this paper, we study the lattice theoretic properties of the integer points in $\mathrm{SCR}(D)$. Given a rational linear subspace $S \subseteq \mathbb{R}^A$, an *integral basis for* S is a subset $B \subseteq S \cap \mathbb{Z}^A$ of linearly independent vectors such that every vector in $S \cap \mathbb{Z}^A$ is an integral linear combination of B.

Theorem 1.1. *Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. Let $\mathcal{F}$ be a nonempty family over ground set V such that $\emptyset, V \notin \mathcal{F}$, and the following face of $\mathrm{SCR}(D)$ is nonempty:*

$$F := \mathrm{SCR}(D) \cap \left\{ x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|, \forall U \in \mathcal{F} \right\}.$$

Suppose $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$. *Then $F \cap \{0,1\}^A$ contains an integral basis for* $\mathrm{lin}(F)$.

Above, $\mathrm{lin}(F)$ refers to the linear hull of F. It can be readily checked that the GCD condition is necessary for $F \cap \{0,1\}^A$ to contain an integral basis for $\mathrm{lin}(F)$. Theorem 1.1 is a consequence of a more general theorem about the lattice generated by the integer points in any face of $\mathrm{SCR}(D)$ where the GCD condition is replaced by '$1 - |\delta^-(U)| \neq 0$ for some $U \in \mathcal{F}$'. This theorem is stated in §6.2.

Theorem 1.1 is best possible in two different ways. First, the result does not extend to faces F involving both $\mathbf{0} \leq x \leq \mathbf{1}$ and (CUT) inequalities. Secondly, for the face F from Theorem 1.1, the integer points in F do not necessarily form a *Hilbert basis*, so the result cannot be strengthened in this direction either. We shall explain both of these points further in §6.7.

1.1 Three applications of the main theorem

Let us discuss some applications of our result.

Woodall's conjecture. Let $D = (V, A)$ be a digraph whose underlying undirected graph is connected. A *dicut* is the set of arcs leaving a nonempty proper vertex subset with no incoming arc, i.e., it is of the form $\delta^+(U)$ where $U \subset V, U \neq \emptyset$ and $\delta^-(U) = \emptyset$. A *dijoin* is an arc subset whose contraction makes the digraph strongly connected. Subsequently, every strengthening set is a dijoin. It can be readily checked that J is a dijoin if, and only if, J intersects every dicut at least once.

A famous conjecture by Douglas Woodall states that the maximum number of disjoint dijoins is equal to the minimum size of a dicut [23]. This conjecture has a convenient reformulation that appears in an unpublished note by Lex Schrijver.

Conjecture 1.2 ([20])**.** *Let $\tau \geq 2$ be an integer, and let $D = (V, A)$ be a digraph, where every dicut has size at least τ. Then A can be partitioned into τ strengthening sets.*

Note the difference between the original formulation of Woodall's conjecture and Conjecture 1.2. While the former is concerned with *packing* dijoins, the latter seeks a *partition* into strengthening sets. This subtle difference comes from the key distinction that while every superset of a dijoin is also a dijoin, a superset of a strengthening set may not necessarily remain a strengthening set.

As a consequence of Theorem 1.1, we obtain the following relaxation of this conjecture. For a subset $J \subseteq A$, denote by $\mathbf{1}_J \in \{0,1\}^A$ the indicator vector of J.

Theorem 1.3. *Let $\tau \geq 2$ be an integer, and let $D = (V, A)$ be a digraph where the minimum size of a dicut is τ. Then there exists an assignment $\lambda_J \in \mathbb{Z}$ to every strengthening set J that intersects every minimum dicut exactly once, such that $\sum_J \lambda_J \mathbf{1}_J = \mathbf{1}$, $\mathbf{1}^\top \lambda = \tau$, and $\left\{\mathbf{1}_J : \lambda_J \neq 0\right\}$ is an integral basis for its linear hull.*

Observe that Conjecture 1.2 states that one can replace $\lambda_J \in \mathbb{Z}$ by $\lambda_J \in \mathbb{Z}_{\geq 0}$ in this theorem. This result does not extend to the capacitated setting; we shall explain this in §6.7.

p-adic programming. Given a prime number $p \geq 2$, a rational number is *(finitely) p-adic* if it is of the form r/p^k for some integer r and nonnegative integer k, and a vector is *p-adic* if each entry is a p-adic rational number. The 2-adic, or *dyadic*, rationals are important for numerical computations because they have a finite binary representation, and therefore can be represented exactly on a computer in floating-point arithmetic. Recently, the first two authors along with Guenin and Tunçel characterized when a linear program admits an optimal solution that is p-adic, and furthermore, they provided a polynomial algorithm for solving a linear program whose domain is restricted to the set of p-adic vectors [1].

Theorem 1.1 implies, for any prime number $p \geq 2$, the existence of a sparse p-adic optimal solution to a linear program related to packing dijoins. To elaborate, let $D = (V, A)$ be a digraph whose underlying undirected graph is connected. Denote by M the matrix whose columns are labeled by A, and whose rows are the indicator vectors of the dijoins of D. Consider the following pair of dual linear programs,

$$(P) \quad \min\left\{\mathbf{1}^\top x : Mx \geq \mathbf{1}, x \geq \mathbf{0}\right\} \qquad\qquad (D) \quad \max\left\{\mathbf{1}^\top y : M^\top y \leq \mathbf{1}, y \geq \mathbf{0}\right\}$$

where $\mathbf{1}, \mathbf{0}$ denote the all-ones and all-zeros vectors of appropriate dimensions, respectively. A seminal theorem is that the primal linear program (P) models exactly the *minimum dicut problem*, i.e., (P) admits an integral optimal solution ([17], see [4], §1.3.4). Woodall's conjecture equivalently states that the dual linear program (D), in turn, computes the maximum number of pairwise disjoint dijoins, that is, (D) admits an integral optimal solution [23]. The main result of this paper implies some number-theoretic evidence for this conjecture, as it has the following consequence.

Theorem 1.4. *For any prime number $p \geq 2$, (D) admits a p-adic optimal solution with at most $2|A|$ nonzero entries.*

Observe that Carathéodory's theorem guarantees an optimal solution to (D) with at most $|A|$ nonzero entries. Theorem 1.4 guarantees a p-adic optimal solution to (D), all the while losing only a factor 2 in the guarantee for the number of nonzero entries.

This theorem does not extend to the capacitated setting. More specifically, if the objective function of (P) is replaced by $c^\top x$ for a nonnegative integral vector c, then (D) may not necessarily have a p-adic optimal solution, for any prime number $p \neq 2$, as we shall explain in §6.7. Interestingly, it has very recently been shown that (D) always admits a dyadic optimal solution in the capacitated setting [14]; the techniques do not seem to yield a guarantee on the number of nonzero entries of a solution.

Hypergraph orientations. Let $H = (V, \mathcal{E})$ be a hypergraph. An *orientation of* H consists in designating to each hyperedge $E \in \mathcal{E}$ a node inside as the *head* of E, i.e., it is a mapping $O : \mathcal{E} \to V$ such that $O(E) \in E$ for each $E \in \mathcal{E}$. The orientation is *strongly connected* if for each $X \subset V, X \neq \emptyset$, there exists a hyperedge whose designated head is inside X, and has at least one node outside X.

Two orientations of H are *head-disjoint* if no hyperedge has the same head in both orientations. It is well-known that a graph, which is simply a 2-uniform hypergraph, has 2 head-disjoint strongly connected orientations if, and only if, the graph is 2-edge-connected. The following unpublished conjecture by Bérczi and Chandrasekaran attempts to extend one direction of this to general τ-uniform hypergraphs.

Given two subsets $X, E \subseteq V$, we say that X *separates* E if $E \cap X \neq \emptyset$ and $E \not\subseteq X$. For $X \subseteq V$, denote by $d_H(X)$ the sum of $|X \cap E|$ ranging over all hyperedges $E \in \mathcal{E}$ separated by X.

Conjecture 1.5. *Let $H = (V, \mathcal{E})$ be a τ-uniform hypergraph such that $d_H(X) \geq \tau$ for all $X \subset V, X \neq \emptyset$. Then H has τ pairwise head-disjoint strongly connected orientations.*

For $\tau = 3$, a weaker form of this conjecture appears explicitly in ([11], Conjecture 9.4.15). We prove the following relaxation of this conjecture.

Theorem 1.6. *Let $\tau \geq 2$ be an integer, and let $H = (V, \mathcal{E})$ be a τ-uniform hypergraph such that $d_H(X) \geq \tau$ for all $X \subset V, X \neq \emptyset$. Then there exists an assignment $\lambda_O \in \mathbb{Z}$ to every strongly connected orientation $O : \mathcal{E} \to V$ such that*

$$\sum_{O(E)=v} \lambda_O = 1 \qquad \forall E \in \mathcal{E}, \forall v \in E,$$

and $|\{O : \lambda_O \neq 0\}| \leq (\tau - 1)|\mathcal{E}| + 1$.

Note that Conjecture 1.5 states that one can replace $\lambda_O \in \mathbb{Z}$ by $\lambda_O \in \mathbb{Z}_{\geq 0}$ above.

1.2 The dijoin polyhedron and digrafts

Theorem 1.1 is a consequence of a lattice theoretic result about the dijoin polyhedron of bipartite digraphs. To this end, for a digraph $D = (V, A)$, let

$$\mathrm{DIJ}(D) := \left\{ x \in \mathbb{R}^A : x(\delta^+(U)) \geq 1, \,\forall \text{ dicut } \delta^+(U); x \geq \mathbf{0} \right\}.$$

It is known that $\mathrm{DIJ}(D)$ is an integral polyhedron, and its vertices are precisely the indicator vectors of the (inclusionwise) minimal dijoins of D ([17], see [4], §1.3.4).

Definition 1.7 (bipartite digraph)**.** *A digraph is* bipartite *if every node is a source or a sink.*

Recently, the first two authors and Zlatin demonstrated the importance of bipartite digraphs in studying Woodall's conjecture, by making steps towards the problem by first reducing the conjecture to a special class of bipartite digraphs [3]. We shall follow these footsteps by studying faces of the dijoin polyhedron of a bipartite digraph.

Definition 1.8 (digraft)**.** *A* digraft *is a pair $(D = (V, A), \mathcal{F})$ where D is a bipartite digraph, the underlying undirected graph of D is 2-edge-connected, and $\mathcal{F}$ is a family over ground set V such that (a) $\emptyset, V \notin \mathcal{F}$, (b) if $U \in \mathcal{F}$ then $\delta^-(U) = \emptyset$, (c) $V \setminus v \in \mathcal{F}$ for every sink v of D, and (d) the following face of* $\mathrm{DIJ}(D)$ *is nonempty:*

$$F(D, \mathcal{F}) := \mathrm{DIJ}(D) \cap \left\{ x \in \mathbb{R}^A : x(\delta^+(U)) = 1, \,\forall U \in \mathcal{F} \right\} \subseteq [0, 1]^A.$$

The choice of the 'digraft' terminology mirrors that of a 'graft', an object that shows up in the context of the *minimum T-join problem*, and is loosely related to the *minimum dijoin problem* (see [4], §1.3.5).

Theorem 1.1 is a consequence of the following result.

Theorem 1.9. *Let $(D = (V, A), \mathcal{F})$ be a digraft. Then $F(D, \mathcal{F}) \cap \{0, 1\}^A$ contains an integral basis for $\text{lin}(F(D, \mathcal{F}))$.*

It may not be clear how Theorem 1.9 is related to Theorem 1.1. To de-mystify this connection, let $(D = (V, A), \mathcal{F})$ be a digraft. As the underlying undirected graph of D is 2-edge-connected, every minimal dijoin is a strengthening set (see [22], Theorem 55.1). Furthermore, every strengthening set that has exactly one arc incident with every sink, is also a minimal dijoin. Subsequently, $F(D, \mathcal{F}) = \text{SCR}(D) \cap \{x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|, \forall U \in \mathcal{F}\}$. Furthermore, $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$. Thus, Theorem 1.9 follows from Theorem 1.1. The converse implication also holds, though we save the proof of this for §6 after we prove Theorem 1.9 directly.

We find Theorem 1.9 more convenient to work with than Theorem 1.1. At the highest level, one explanation for this is that every point in $F(D, \mathcal{F}) \cap \{0, 1\}^A$ is the indicator vector of an arc subset J that has degree one at every sink of D, and degree at least one at every source of D; so that J may be viewed as a perfect b_J-matching in a bipartite graph, for some degree vector b_J. Fixing the degree of J at each sink to one has advantages: first, the cardinality of J becomes invariant and equal to the number of sinks of D; secondly, in this case, J is a minimal dijoin if and only if it is a strengthening set, an equivalence which we utilized above.

1.3 Proof overview of Theorem 1.9

Let us provide an overview of the proof of Theorem 1.9. To this end, let $(D = (V, A), \mathcal{F})$ be a digraft. Our goal is to find an integral basis in $F(D, \mathcal{F}) \cap \{0, 1\}^A$ for $\text{lin}(F(D, \mathcal{F}))$. The following is an important notion needed in the proof.

Definition 1.10 (basic digraft)**.** *A digraft $(D = (V, A), \mathcal{F})$ is* basic *if whenever $F(D, \mathcal{F}) \subseteq \{x : x(\delta^+(U)) = 1\}$ for some dicut $\delta^+(U)$, then $U \in \mathcal{F}$, and $|U| \in \{1, |V| - 1\}$.*

The proof proceeds by first decomposing the digraft into basic pieces along dicuts $\delta^+(U)$ such that $1 < |U| < |V| - 1$ and $F(D, \mathcal{F}) \subseteq \{x : x(\delta^+(U)) = 1\}$. Once we find integral bases for the basic pieces, then by composing the bases together in a natural manner, we obtain an integral basis in $F(D, \mathcal{F}) \cap \{0, 1\}^A$.

The challenge now is to find an integral basis for a basic digraft. Here comes a key idea of the proof, which is to study the facet-defining inequalities of $F(D, \mathcal{F})$.

Definition 1.11 (basic robust digraft)**.** *A basic digraft $(D = (V, A), \mathcal{F})$ is* robust *if every facet-defining inequality for $F(D, \mathcal{F})$ is equivalent to $x_a \geq 0, a \in A$, or $x(\delta^+(u)) \geq 1$ for some source u of D.*

We then divide the proof into two cases, depending on whether the basic digraft is robust.

Basic robust digrafts. For a basic robust digraft $(D, \mathcal{F})$, we prove that $F(D, \mathcal{F})$ is a very special polyhedron. To elaborate, for a polyhedron $P \subseteq \mathbb{R}^n$ and $k \geq 0$, define kP as the set of all points of the form $\sum_{p \in P} \lambda_p p$ where $\lambda \in \mathbb{R}^P_{\geq 0}$ and $\mathbf{1}^\top \lambda = k$. P has the *integer decomposition property* if for every integer $k \geq 1$, every integral point in kP can be written as the sum of k integral points in P. The inequality description of $F(D, \mathcal{F})$, along with a classic result of de Werra [7] on balanced edge-colourings of bipartite graphs, allows for the following theorem.

Theorem 1.12. *Let $(D = (V, A), \mathcal{F})$ be a basic robust digraft. Then $F(D, \mathcal{F})$ has the integer decomposition property, and $\mathrm{aff}(F(D, \mathcal{F})) = \{x : Mx = \mathbf{1}\}$ for some $M \in \mathbb{Z}^{m\times n}$ with $m \geq 1$.*

Theorem 1.9 for basic robust digrafts now follows from the following general-purpose result about polyhedra with the integer decomposition property.

Theorem 1.13. *Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron with the integer decomposition property, where $\mathrm{aff}(P) = \{x : Ax = b\}$ for $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ such that $m \geq 1$, $b \neq \mathbf{0}$, and $\gcd\{b_i : i \in [m]\} = 1$. Then $P \cap \mathbb{Z}^n$ contains an integral basis for $\mathrm{lin}(P)$.*

Theorem 1.13 is obtained by first proving that $P \cap \mathbb{Z}^n$ forms an *integral generating set for a cone*, better known as a *Hilbert basis*, and then using a result of Gerards and Sebő [13] about such sets to finish the proof.

Basic non-robust digrafts. In the remaining case, where the basic digraft $(D = (V, A), \mathcal{F})$ is not robust, $F(D, \mathcal{F})$ has a facet-defining dicut inequality $x(\delta^+(U)) \geq 1$ that is not equivalent to $x(\delta^+(u)) \geq 1$ for any source u. We decompose the digraft into two pieces along the dicut $\delta^+(U)$, called the '$(U, V \setminus U)$-contractions' of $(D, \mathcal{F})$. Each of the two $(U, V \setminus U)$-contractions is again a digraft, so by induction, we may pick integral bases B_1, B_2 for the two pieces, and compose them in a natural way to obtain a linearly independent set $B' \subseteq F(D, \mathcal{F}) \cap \{0, 1\}^A$. However, there are two key challenges to turn B' into an integral basis B. First, B' is at least one vector away from forming a linear basis for $\mathrm{lin}(F(D, \mathcal{F}))$, and a priori, we do not know the number of extra vectors we would need to add. Secondly, a linear basis is a long way from an integral one, so we need to extend B' very carefully. We have two lemmas that address these issues.

The first issue stems from the fact that the two $(U, V \setminus U)$-contractions are not necessarily basic digrafts. This can be fatal as we could lose our guarantee on the size of $|B \setminus B'|$. However, we will be able to prove that both of these pieces share a key property with basic digrafts, thus allowing us to guarantee that $|B \setminus B'| = 1$. To describe this property, we need a couple of definitions.

Definition 1.14 (tight and active nodes)**.** *Let $(D = (V, A), \mathcal{F})$ be a digraft. A node $u \in V$ is* tight *for the digraft if $F(D, \mathcal{F}) \subseteq \{x : x(\delta(v)) = 1\}$; the node is* active *for the digraft if it is not tight.*

Note that all sinks of a digraft are tight, i.e., every active node is a source. Note further that while singletons and complements of singletons in $\mathcal{F}$ give rise to tight nodes, there may be more (implied) tight nodes.

Definition 1.15 (affine critical digraft)**.** *Let $(D = (V, A), \mathcal{F})$ be a digraft, and let V^t be the set of tight nodes. The digraft is* affine critical *if $\mathrm{aff}(F(D, \mathcal{F})) = \left\{x : x(\delta(v)) = 1, \forall v \in V^t\right\}$.*

The following important lemma addresses the first issue mentioned above.

Lemma 1.16 (Affine Critical Lemma)**.** *Let $(D = (V, A), \mathcal{F})$ be a basic digraft that is not robust. Let $x(\delta^+(U)) \geq 1$ be a facet-defining dicut inequality for $F(D, \mathcal{F})$ that is not equivalent to $x(\delta^+(u)) \geq 1$ for any active source u. Then $(D, \mathcal{F})$ and its $(U, V \setminus U)$-contractions are affine critical digrafts each of which contains at least one active source. Furthermore, for $i \in \{1, 2\}$, every active source for $(D, \mathcal{F})$ in U_{3-i} is an active source for $(D_i, \mathcal{F}_i)$, and vice versa.*

This lemma is a byproduct of a careful analysis of the dimension of $F(D,\mathcal{F})$, and the study of a characteristic quantity of a digraft that we call the *slack*.

The Affine Critical Lemma guarantees that to turn B' into a linear basis for $\mathrm{lin}(F(D,\mathcal{F}))$, we just need to add one more vector b from $F(D,\mathcal{F})\cap\{0,1\}^A$, which must inevitably satisfy $b(\delta^+(U))>1$. As it turns out, integrality of the basis can be guaranteed if $b(\delta^+(U))=2$, whose existence will be guaranteed by the following lemma, thus addressing the second issue. This lemma is ultimately enabled by the Exchange Axiom for M-convex sets.

Lemma 1.17 (Jump-Free Lemma). *Let $(D=(V,A),\mathcal{F})$ be a digraft, let $\delta^+(U)$ be a dicut, and let $x,y\in F(D,\mathcal{F})\cap\{0,1\}^A$ where $\lambda_1:=x(\delta^+(U))<y(\delta^+(U))=:\lambda_2$. Then for any integer $\lambda\in(\lambda_1,\lambda_2)$, there exists $z\in F(D,\mathcal{F})\cap\{0,1\}^A$ such that $z(\delta^+(U))=\lambda$.*

Outline of the paper. We start off in §2 by studying the integer decomposition property, and proving Theorem 1.13. In §3, we discuss the exchange axiom for M-convex sets, and use it to prove the Jump-Free Lemma. In §4, we introduce the 'slack' of a digraft, study the dimension of the faces of the dijoin polyhedron, and then prove the Affine Critical Lemma. These three sections can be read independently from one another. Theorem 1.13 and Theorem 1.9 are then proved in §5. The final section §6 is dedicated to proving Theorem 1.1 and its three applications.

As a last note, it should be acknowledged that our work is heavily inspired by the works of Edmonds, Lovász, and Pulleyblank [10], Lovász [15], and de Carvalho, Lucchesi, and Murty [6] on the matching lattice of a matching-covered graph.

2 IGSCs and the integer decomposition property

A finite set $H\subseteq\mathbb{Z}^n$ is an *integral generating set for a subspace (IGSS)* if every integral vector in $\mathrm{lin}(H)$ can be written as an integer linear combination of the vectors in H. A finite set $H\subseteq\mathbb{Z}^n$ is an *integral generating set for a cone (IGSC)* if every integral vector in $\mathrm{cone}(H)$ can be written as an integral conic combination of the vectors in H ([2], §7).[1] It can be readily checked that every IGSC is also an IGSS (see [2], §7). We have the following theorem, which is essentially due to Gerards and Sebő [13].

Theorem 2.1. *Let $H\subseteq\mathbb{Z}^n$ be an IGSC such that $\mathrm{cone}(H)$ is pointed. Then H contains an integral basis for $\mathrm{lin}(H)$.*

Proof. Let $H\subseteq\mathbb{Z}^n$ be an IGSC such that $\mathrm{cone}(H)$ is pointed. Let r be the rank of H. It follows from ([13], (2)) that there exist linearly independent vectors $h_1,\dots,h_r$ in H such that $B:=\{h_1,\dots,h_r\}$ is an IGSC, hence an IGSS. Thus, B is an integral basis, as required. □

Note that the condition that $\mathrm{cone}(H)$ is pointed is necessary. For example, $\{-2,3\}$ forms an IGSC, but it does not contain an integral basis for $\mathbb{R}$.

Let $g\in\mathbb{Z}_{\geq 1}$. A finite set $H\subseteq\mathbb{Z}^n$ is a *$\frac{1}{g}$-integral generating set for a cone ($\frac{1}{g}$-IGSC)* if every integral vector in $\mathrm{cone}(H)$ can be written as a $\frac{1}{g}$-integral conic combination of the vectors in H.

[1] Sometimes H is referred to as a *Hilbert basis*, but we refrain from using this terminology as it can be confusing.

Theorem 2.2. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron with the integer decomposition property, where* $\mathrm{aff}(P) = \{x : Ax = b\}$ *for some $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m \setminus \mathbf{0}$ with $m \geq 1$. Let $g := \gcd\{b_i : i \in [m]\}$. Then $P \cap \mathbb{Z}^n$ is a $\frac{1}{g}$-IGSC.*

Proof. Let $\bar{x}$ be an integral vector in $\mathrm{cone}(P \cap \mathbb{Z}^n)$, so $\bar{x} = \sum_{p\in P\cap\mathbb{Z}^n} \lambda_p p$ for some assignment $\lambda_p \in \mathbb{R}_{\geq 0}$ to every $p \in P \cap \mathbb{Z}^n$, where only finitely many λ_p's are nonzero. Let $k := \mathbf{1}^\top \lambda \geq 0$. Note that $\bar{x} \in kP \cap \mathbb{Z}^n$. We claim that $gk \in \mathbb{Z}$. To see this, note that

$$A\bar{x} = \sum_{p\in P\cap\mathbb{Z}^n} \lambda_p Ap = \sum_{p\in P\cap\mathbb{Z}^n} \lambda_p b = kb.$$

Given that both $A, \bar{x}$ are integral, it follows that $A\bar{x} = kb$ is also integral, so $k \in \frac{1}{g}\mathbb{Z}$.

If $k = 0$, then $\bar{x} = 0$. Otherwise, $gk \geq 1$. As $g\bar{x} \in gkP \cap \mathbb{Z}^n$ and P has the integer decomposition property, it follows that $g\bar{x}$ can be written as the sum of gk points in $P \cap \mathbb{Z}^n$. In both cases, we expressed $\bar{x}$ as a $\frac{1}{g}$-integral conic combination of the vectors in $P \cap \mathbb{Z}^n$, thus finishing the proof. □

We are now ready to prove Theorem 1.13.

Proof of Theorem 1.13. Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron with the integer decomposition property, where $\mathrm{aff}(P) = \{x : Ax = b\}$ for $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ such that $m \geq 1$, $b \neq \mathbf{0}$, and $\gcd\{b_i : i \in [m]\} = 1$. It follows from Theorem 2.2 for $g = 1$ that $P \cap \mathbb{Z}^n$ is an IGSC. As P is pointed, so is $\mathrm{cone}(P \cap \mathbb{Z}^n)$, so by Theorem 2.1, $P \cap \mathbb{Z}^n$ contains an integral basis for $\mathrm{lin}(P \cap \mathbb{Z}^n) = \mathrm{lin}(P)$, where this equality follows from the integrality of P, as required. □

3 M-convex sets and the Jump-Free Lemma

Let $(D = (V, A), \mathcal{F})$ be a digraft. In this section, we will see an affine function which maps $F(D, \mathcal{F}) \subseteq \mathbb{R}^A$ to a base polyhedron in $\mathbb{R}^V$, and $F(D, \mathcal{F}) \cap \{0, 1\}^A$ to an 'M-convex set'.[2] The theory of perfect b-matchings in bipartite graphs allows us to construct a (not necessarily unique) inverse to this function. This inverse map, together with the 'Exchange Axiom' for M-convex sets leads to a proof of the Jump-Free Lemma. We will also state and prove a result, more specifically Theorem 3.2, which will be needed in the next section.

For a vertex subset $U \subseteq V$, denote by $\mathrm{sources}(U)$ and $\mathrm{sinks}(U)$ the sets of sources and sinks in U, respectively, and define the *discrepancy of* U as $\mathrm{disc}(U) := |\mathrm{sinks}(U)| - |\mathrm{sources}(U)|$ [3]. Let $\mathcal{U} := \{U \subset V : U \neq \emptyset, \delta^-(U) = \emptyset\}$, which is a *crossing family* over ground set V, meaning that $U \cap W, U \cup W \in \mathcal{U}$ for all pairs $U, W \in \mathcal{U}$ that *cross*, i.e., $U \cap W \neq \emptyset$ and $U \cup W \neq V$. The function $\mathrm{disc} : \mathcal{U} \to \mathbb{Z}$ forms a *crossing supermodular function*, meaning that $\mathrm{disc}(U \cap W) + \mathrm{disc}(U \cup W) \geq \mathrm{disc}(U) + \mathrm{disc}(W)$ for all pairs $U, W \in \mathcal{U}$ that cross. (In fact, equality holds here, but all we need is the inequality.)

Consider the polytope

$$P(D) := \left\{z \in \mathbb{R}^V : z(U) \geq 1 + \mathrm{disc}(U), \forall U \in \mathcal{U};\ z(V) = \mathrm{disc}(V)\right\}.$$

Above, we have an inequality for every set U in the crossing family $\mathcal{U}$, and an equality constraint for the ground set V. Given that the right-hand side values of the inequalities $U \mapsto 1 + \mathrm{disc}(U)$

[2] The terminology in this section relating to discrete convex analysis follows [19].

form a crossing supermodular function over the crossing family $\mathcal{U}$, and $P(D) \neq \emptyset$ as we shall see shortly, we thus obtain that $P(D)$ is an integral *base polyhedron* (a.k.a. a polymatroid), by a result of Fujishige [12]. It can be readily checked that for all $z \in P(D)$, we have $z_u \geq 0 \geq z_v$ for every source u and sink v of D.

Let

$$P(D, \mathcal{F}) := P(D) \cap \{z : z_v = 0, \forall v \in \text{sinks}(V);\ z(U) = 1 + \text{disc}(U), \forall U \in \mathcal{F}\}.$$

As a face of an integral base polyhedron, $P(D, \mathcal{F})$ is also an integral base polyhedron. Subsequently, $P(D, \mathcal{F}) \cap \mathbb{Z}^V$ is an *M-convex set*, that is, it possesses the following property:

> *Exchange Axiom:* For $z, t \in P(D, \mathcal{F}) \cap \mathbb{Z}^V$ and $u \in \text{supp}^+(z - t)$, there exists $v \in \text{supp}^-(z - t)$ such that $z' := z - \mathbf{1}_u + \mathbf{1}_v \in P(D, \mathcal{F}) \cap \mathbb{Z}^V$. We say that z' is obtained by the exchange pair (u, v) for (z, t).

Above, $\text{supp}^+(z) = \{v : z_v > 0\}$ and $\text{supp}^-(z) = \{v : z_v < 0\}$. For more on base polyhedra and M-convex sets, we refer the reader to Murota's excellent book on Discrete Convex Analysis ([19], Chapter 4).

Let $x \in \mathbb{R}^A$ such that $x(\delta(v)) = 1$ for every sink v. For each $v \in V$, let $z_v := x(\delta(v)) - 1$. Then $z_v = 0$ for every sink v, and $z(U) = x(\delta^+(U)) + \text{disc}(U)$ for every dicut $\delta^+(U)$. Subsequently, $x \in F(D, \mathcal{F})$ if and only if $z \in P(D, \mathcal{F})$. In particular, given that $F(D, \mathcal{F}) \neq \emptyset$, we conclude that $P(D, \mathcal{F}) \neq \emptyset$, and so $P(D) \neq \emptyset$, as promised.

We just showed how to map every (integral) point in $F(D, \mathcal{F})$ to an (integral) point in $P(D, \mathcal{F})$. Below we show that it is possible to go in reverse; only the first part of the lemma is needed for the proof of the Jump-Free Lemma, while the second (stronger) part is needed later.

Lemma 3.1. *Let $(D = (V, A), \mathcal{F})$ be a digraft, and let $z \in P(D, \mathcal{F}) \cap \mathbb{Z}^V$.*

1. *There exists $J \subseteq A$ such that $|J \cap \delta(v)| - 1 = z_v$ for each $v \in V$.*
2. *For each $a \in A$, there exists $J \subseteq A$ such that $a \in J$ and $|J \cap \delta(v)| - 1 = z_v$ for each $v \in V$.*

Proof. Let $b := \mathbf{1} + z \in \mathbb{Z}^V$. Note that $b_v = 1$ for every sink v, and $b_u \geq 1$ for every source u.

(1) This part asks for an arc subset J such that $|J \cap \delta(v)| = b_v$. A *perfect b-matching* is a vector $x \in \mathbb{Z}^A_{\geq 0}$ such that $x(\delta(v)) = b_v$ for every vertex v. Given that $b_v = 1$ for every sink v, and $x \geq \mathbf{0}$, it follows that every perfect b-matching is a $0,1$ vector. Thus, to prove this part, it suffices to argue the existence of a perfect b-matching.

To this end, denote by S and T the sets of sources and sinks of D, respectively. It is known that a perfect b-matching exists if, and only if, $b(S) = b(T)$, and for $b(U \cap S) - b(U \cap T) \geq 0$ for every dicut $\delta^+(U)$ ([22], Corollary 21.1b, see [3], Theorem 4.10).

Given that $b_v = 1$ for every sink v, the equality $z(V) = \text{disc}(V)$ is equivalent to $b(S) = b(T)$, while the inequality $z(U) \geq 1 + \text{disc}(U)$ is equivalent to $b(U \cap S) - b(U \cap T) \geq 1$ for every dicut $\delta^+(U)$. In particular, there exists a perfect b-matching, as required.

(2) Suppose a has tail u and head w. Let $D' := D \setminus w$, define $z' \in \mathbb{Z}^{V \setminus w}$ as $z'_u := z_u - 1$ and $z'_v := z_v$ for all $v \in V \setminus w \setminus u$, and let $b' := \mathbf{1} + z' \in \mathbb{Z}^{V \setminus w}_{\geq 0}$. We claim that D' has a perfect b'-matching. To this end, note first that $b'(S) = b(S) - 1 = b(T \setminus w) = b'(T \setminus w)$. Furthermore, let $\delta^+_{D'}(U)$ be a dicut of D' for some $U \subset V \setminus w, U \neq \emptyset$. Observe that $\delta^+_D(U)$ is a dicut of D, so by the previous part, $b(U \cap S) - b(U \cap T) \geq 1$. Subsequently,

$$b'(U \cap S) - b'(U \cap (T \setminus w)) \geq (b(U \cap S) - 1) - b'(U \cap (T \setminus w)) = b(U \cap S) - 1 - b(U \cap T) \geq 0.$$

As this inequality holds for every dicut $\delta^+_{D'}(U)$, D' has a perfect b'-matching, as claimed. Adding arc a, we obtain a perfect b-matching x in D such that $x_a = 1$, thereby proving this part. □

We are now ready to prove the Jump-Free Lemma.

Proof of Lemma 1.17. Let $(D = (V, A), \mathcal{F})$ be a digraft, let $\delta^+(U)$ be a dicut, and let $J_1, J_2 \subseteq A$ be subsets such that $\mathbf{1}_{J_1}, \mathbf{1}_{J_2} \in F(D, \mathcal{F})$ and $\lambda_1 := |J_1 \cap \delta^+(U)| < |J_2 \cap \delta^+(U)| =: \lambda_2$. Our goal is to prove that for any integer $\lambda \in (\lambda_1, \lambda_2)$, there exists a $J \subseteq A$ such that $\mathbf{1}_J \in F(D, \mathcal{F})$ and $|J \cap \delta^+(U)| = \lambda$.

For each $i \in \{1, 2\}$, and vertex u, let $z^i_u := |J_i \cap \delta(u)| - 1$. Then $z^1, z^2 \in P(D, \mathcal{F})$. By the Exchange Axiom for the M-convex set $P(D, \mathcal{F}) \cap \mathbb{Z}^V$, there exists a sequence of points $z^1 =: t^1, t^2, \dots, t^k := z^2$ in $P(D, \mathcal{F}) \cap \mathbb{Z}^V$ such that t^{i+1} is obtained by an exchange pair (u^i, v^i) for (t^i, t^k), for each $1 \leq i \leq k - 1$. Note that $k - 1$ is precisely the sum of the nonnegative entries of $z^1 - z^2$. Note further that for each $1 \leq i \leq k - 1$,

$$t^i(U) - t^{i+1}(U) = \mathbf{1}_{u_i}(U) - \mathbf{1}_{v_i}(U) \in \{-1, 0, 1\},$$

thus the set $\{t^i(U) : i = 1, \dots, k\}$ contains all the integers between $z^1(U)$ and $z^2(U)$. Subsequently, given that

$$z^1(U) - \mathrm{disc}(U) = \lambda_1 < \lambda_2 = z^2(U) - \mathrm{disc}(U),$$

we have

$$\{t^i(U) - \mathrm{disc}(U) : i = 1, \dots, k\} \supseteq [\lambda_1, \lambda_2] \cap \mathbb{Z}.$$

Now pick an integer $\lambda \in (\lambda_1, \lambda_2)$, and a t^i such that $t^i(U) - \mathrm{disc}(U) = \lambda$. By Lemma 3.1 (1), there exists a $J \subseteq A$ such that $|J \cap \delta(v)| - 1 = t^i_v$ for each $v \in V$, and so in particular, $\mathbf{1}_J \in F(D, \mathcal{F})$. This is the desired set J, because $|J \cap \delta^+(U)| = t^i(U) - \mathrm{disc}(U) = \lambda$. □

Lemma 3.1 (2) has the following consequence, which will be useful in the next section.

Theorem 3.2. *Let $(D = (V, A), \mathcal{F})$ be a digraft. Then, for every $a \in A$, there exists $J \subseteq A$ such that $a \in J$ and $\mathbf{1}_J \in F(D, \mathcal{F})$.*

Proof. Pick an arbitrary point $z \in P(D, \mathcal{F}) \cap \mathbb{Z}^V$, and let $a \in A$. Then by Lemma 3.1 (2), there exists $J \subseteq A$ such that $a \in J$ and $|J \cap \delta(v)| - 1 = z_v$ for each $v \in V$. As $z \in P(D, \mathcal{F})$, it follows that $\mathbf{1}_J \in F(D, \mathcal{F})$, as required. □

4 The slack, dicut contraction, and the Affine Critical Lemma

In the introduction, we motivated digrafts as a means to describe faces of $\mathrm{DIJ}(D)$ for a bipartite digraph D obtained by setting some dicut inequalities to equality. In this section, we first count the dimension of the face $F(D, \mathcal{F})$ for a digraft $(D, \mathcal{F})$, and in the process, define the notion of the ‘slack’, a novel and characteristic quantity associated with a digraft. When counting the dimension, we must study the equations that define the affine hull of $F(D,\mathcal{F})$. It is not clear whether some non-negativity constraints used to define $\mathrm{DIJ}(D)$, are forced to equality in $F(D,\mathcal{F})$. The following immediate corollary of Theorem 3.2 shows that this is fortunately not the case.

Corollary 4.1. *$F(D,\mathcal{F}) \nsubseteq \{x : x_a = 0\}$ for any digraft $(D = (V,A),\mathcal{F})$ and any arc $a \in A$.* □

Therefore, the affine hull of $F(D,\mathcal{F})$ can be described solely by setting some dicut inequalities to equality. Roughly speaking, the 'slack' captures the contribution of the 'non-trivial dicut' inequalities in defining the affine hull.

We shall formalize the notion of the slack in §4.1. In §4.2, we show how to 'decompose' a digraft along a 'contractible' dicut. In §4.3, we study for a basic digraft the facet-defining inequalities of $F(D,\mathcal{F})$ corresponding to contractible dicuts, and prove crucially that decomposition along this dicut does not change the slack. We are then well-equipped to prove the Affine Critical Lemma in §4.4.

4.1 The slack

Let $(D = (V,A),\mathcal{F})$ be a digraft, and let V^t be the set of tight nodes. Let $F := F(D,\mathcal{F})$. We wish to lower bound the rank r of the equations that hold for F. Let $\mathrm{aff}(F)$ be the affine hull of F. By Corollary 4.1, $\mathrm{aff}(F)$ can be described by two types of equations: (i) $x(\delta(v)) = 1\, \forall v \in V^t$, and (ii) $x(\delta^+(U)) = 1$ for some dicuts $\delta^+(U)$ where $|U| \neq 1, |V|-1$. To lower bound r, we need the following remark.

Remark 4.2. *Let $G = (V,E)$ be a connected graph, and let $v^\star \in V$. Then $\mathbf{1}_{\delta(u)}, u \in V \setminus v^\star$ are linearly independent.*

It can be readily checked that

$$r \geq |V^t| - 1, \tag{1}$$

where we have used the fact that the equations $x(\delta(u)) = 1$, $\forall u \in V^t$ have rank at least $|V^t|-1$, by Remark 4.2. Furthermore, as the underlying undirected graph of D is bipartite, these equations have rank exactly $|V^t|-1$ if and only if $V = V^t$.

Definition 4.3 (κ^t)**.** *Denote by $\kappa^t(D,\mathcal{F})$ the indicator variable for the event $V = V^t$, set to 1 if the event occurs, and 0 otherwise.*

As discussed above, the inequality (1) can be strengthened as follows:

$$r \geq |V^t| - \kappa^t(D,\mathcal{F}), \tag{2}$$

where we have subtracted a 1 only if every vertex of $(D,\mathcal{F})$ is tight.

Denote by $\dim(F)$ the dimension of $\mathrm{aff}(F)$. Then $\dim(F) = |A| - r$, so we obtain from (2) that

$$|A| - |V^t| + \kappa^t(D,\mathcal{F}) \geq \dim(F). \tag{3}$$

There is a subtle difference between the dimensions of the affine and linear hulls of F. Note that the dimension of the linear hull of F is $\dim(F)+1$, given that $\mathbf{0} \notin \mathrm{aff}(F)$.

A characteristic quantity associated with a digraft is the slack in (3).

Definition 4.4 (slack)**.** *The* slack *of* $(D = (V, A), \mathcal{F})$ *is*

$$s(D, \mathcal{F}) := |A| - |V^t| + \kappa^t(D, \mathcal{F}) - \dim(F(D, \mathcal{F})) \geq 0,$$

where V^t *denotes the set of tight nodes of* $(D, \mathcal{F})$.

The following is a straightforward but important characterization of when there is no slack.

Remark 4.5. $s(D, \mathcal{F}) = 0$ *if and only if* $\operatorname{aff}(F(D, \mathcal{F}))$ *is described by* $x(\delta(v)) = 1,\ v \in V^t$.

Subsequently, if $s(D, \mathcal{F}) \geq 1$, then the description of $\operatorname{aff}(F(D, \mathcal{F}))$ also sets some 'non-trivial' dicut inequalities to equality.

Definition 4.6 (tight dicut)**.** *A dicut* $\delta^+(U)$ *is* tight *for* $(D = (V, A), \mathcal{F})$ *if* $F(D, \mathcal{F}) \subseteq \{x : x(\delta^+(U)) = 1\}$. *A tight dicut* $\delta^+(U)$ *is* non-trivial *if* $1 < |U| < |V| - 1$, *otherwise it is* trivial*.*

The following theorem is the main result of this subsection.

Theorem 4.7. *Let* $(D = (V, A), \mathcal{F})$ *be a digraft. Then* $s(D, \mathcal{F}) \geq 1$ *if, and only if, there exists a non-trivial tight dicut* $\delta^+(U)$ *where both* $U, V \setminus U$ *contain active sources.*

Proof. Denote by V^t, V^a the sets of tight and active nodes of $(D, \mathcal{F})$.

($\Leftarrow$) Suppose $\delta^+(U)$ is a non-trivial tight dicut of the digraft such that both $U, V \setminus U$ contain active sources. By definition, the points in $F(D, \mathcal{F})$ satisfy the equation $x(\delta^+(U)) = 1$. We claim that this equation is not implied by the equations $x(\delta(v)) = 1,\ \forall v \in V^t$, which were used to obtain (2), thus giving an improvement of 1 to that inequality, which eventually implies $s(D, \mathcal{F}) \geq 1$.

To prove the linear independence of $x(\delta^+(U)) = 1$ from the other equations, it suffices to prove that $\mathbf{1}_{\delta^+(U)}$ is linearly independent of the vectors $\mathbf{1}_{\delta(v)}, v \in V^t$. Note that

$$\mathbf{1}_{\delta^+(U)} = \sum_{u \in \text{sources}(U)} \mathbf{1}_{\delta^+(u)} - \sum_{v \in \text{sinks}(U)} \mathbf{1}_{\delta^-(v)}.$$

After subtracting a linear combination of $\mathbf{1}_{\delta(v)}, v \in V^t$ from $\mathbf{1}_{\delta^+(U)}$, it suffices to prove that the vector $\sum_{u \in \text{sources}(U) \cap V^a} \mathbf{1}_{\delta(u)}$ is linearly independent of $\mathbf{1}_{\delta(v)}, v \in V^t$. Given that both $U, V \setminus U$ contain active sources, it follows that $\text{sources}(U) \cap V^a$ is a nonempty proper subset of V^a. Let $v^\star \in \text{sources}(V \setminus U) \cap V^a$. By Remark 4.2, the vectors $\mathbf{1}_{\delta(v)}, v \in V \setminus v^\star$ are linearly independent, implying in turn that $\sum_{u \in \text{sources}(U) \cap V^a} \mathbf{1}_{\delta(u)}$ is linearly independent of $\mathbf{1}_{\delta(v)}, v \in V^t$.

($\Rightarrow$) Suppose $s(D, \mathcal{F}) \geq 1$, that is, the inequality in (2) is not tight. This implies that there exists a tight dicut $\delta^+(U)$ such that the equation $x(\delta^+(U)) = 1$ is linearly independent of the equations $x(\delta(v)) = 1,\ \forall v \in V^t$. In particular, $\mathbf{1}_{\delta^+(U)}$ is linearly independent of the vectors $\mathbf{1}_{\delta(v)}, v \in V^t$. Given that

$$\mathbf{1}_{\delta^+(U)} = \sum_{u \in \text{sources}(U)} \mathbf{1}_{\delta^+(u)} - \sum_{v \in \text{sinks}(U)} \mathbf{1}_{\delta^-(v)} = \sum_{v \in \text{sinks}(V \setminus U)} \mathbf{1}_{\delta^-(v)} - \sum_{u \in \text{sources}(V \setminus U)} \mathbf{1}_{\delta^+(u)},$$

it follows that both $U, V \setminus U$ must contain active sources. Subsequently, $|U| \neq 1, |V| - 1$, so $\delta^+(U)$ is a non-trivial tight dicut, thereby finishing the proof. $\square$

Let us give an intuitive interpretation of the slack $s := s(D, \mathcal{F})$. The affine hull of $F(D, \mathcal{F})$ is described by two types of constraints: $x(\delta(v)) = 1, \ \forall v \in V^t$, and $x(\delta^+(U)) = 1$ for a non-trivial tight dicut $\delta^+(U)$. The slack s computes the additional contribution of non-trivial tight dicuts —in terms of rank increase— in defining the affine hull. Furthermore, if $s \geq 1$, then there exists a cross-free family of s non-trivial tight dicuts, which can be used to give a 'decomposition' of the digraft into $s+1$ pieces partitioning the active sources of $(D, \mathcal{F})$ into $s+1$ nonempty parts, such that each piece of the form $(D', \mathcal{F}')$ satisfies $s(D', \mathcal{F}') = \kappa^t(D', \mathcal{F}') = 0$. Though these ideas can be formalized, we refrain from doing so here as it is outside the scope of this paper.

4.2 Dicut contractions

Let $(D = (V, A), \mathcal{F})$ be a digraft. In this subsection, we show how to decompose $(D, \mathcal{F})$ along certain dicuts into two smaller digrafts. We need a few preliminaries.

Definition 4.8 (closure)**.** *The* closure of $\mathcal{F}$ *for* $(D, \mathcal{F})$ *is the family of subsets* $U \subset V, U \neq \emptyset$ *such that* $\delta^-(U) = \emptyset$ *and* $F(D, \mathcal{F}) \subseteq \{x : x(\delta^+(U)) = 1\}$.

The following lemma will be useful in this subsection.

Lemma 4.9. *Let* $(D = (V, A), \mathcal{F})$ *be a digraft, and let* $\mathcal{C}$ *be the closure of* $\mathcal{F}$ *for* $(D, \mathcal{F})$*. Then* $\mathcal{F} \subseteq \mathcal{C}$*, and* $\mathcal{C}$ *is a crossing family.*

Proof. The inclusion $\mathcal{F} \subseteq \mathcal{C}$ is clear. Let $U, W \in \mathcal{C}$ cross, and let $x \in F(D, \mathcal{F})$. As $\delta^+(U), \delta^+(W)$ are dicuts, then so are $\delta^+(U \cap W), \delta^+(U \cup W)$. Subsequently, $x(\delta^+(U \cap W)), x(\delta^+(U \cup W)) \geq 1$, and so

$$2 = x(\delta^+(U)) + x(\delta^+(W)) = x(\delta^+(U \cap W)) + x(\delta^+(U \cup W)) \geq 2.$$

Equality must hold throughout, so $x(\delta^+(U \cap W)) = x(\delta^+(U \cup W)) = 1$. As this holds for all $x \in F(D, \mathcal{F})$, it follows that $U \cap W, U \cup W \in \mathcal{C}$. □

Of interest are those 'non-trivial' dicut inequalities that expose a nonempty face of $F(D, \mathcal{F})$.

Definition 4.10 (contractible dicut)**.** *A dicut* $\delta^+(U)$ *is* contractible *if* $1 < |U| < |V| - 1$ *and* $F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\} \neq \emptyset$.

The following operation justifies the choice of the terminology above.

Definition 4.11 ($(U, V \setminus U)$-contractions)**.** *Suppose* $\delta^+(U)$ *is a contractible dicut of* $(D, \mathcal{F})$*. Let* $\overline{\mathcal{F}}$ *be the closure of* $\mathcal{F} \cup \{U\}$ *for the digraft* $(D, \mathcal{F} \cup \{U\})$*. Let* $U_1 := U$ *and* $U_2 := V \setminus U$*. Let* $D_i = (V_i, A_i)$ *be the bipartite digraph obtained from* D *after shrinking* U_i *to a single node* u_i*; so* $V_i = \{u_i\} \cup U_{3-i}$*. Let*

$$\mathcal{F}_i := \{W : W \cap U_i = \emptyset, W \in \overline{\mathcal{F}}\} \cup \{(W \setminus U_i) \cup \{u_i\} : U_i \subseteq W, W \in \overline{\mathcal{F}}\}.$$

We refer to $(D_i, \mathcal{F}_i), i = 1, 2$ *as the* $(U, V \setminus U)$-contractions of $(D, \mathcal{F})$.

Note that u_1 is a source in D_1 and $\{u_1\} \in \mathcal{F}_1$, and u_2 is a sink in D_2 and $V_2 \setminus u_2 = U_1 \in \mathcal{F}_2$.

We now explain how a $(U, V \setminus U)$-contraction decomposes the digraft $(D, \mathcal{F})$, as well as the $0, 1$ points in $F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\}$, into digrafts $(D_i, \mathcal{F}_i)$, and $0, 1$ points in $F(D_i, \mathcal{F}_i)$, respectively. We also explain when and how two $0, 1$ points in $F(D_i, \mathcal{F}_i), i = 1, 2$ can be composed to give a $0, 1$ point in $F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\}$.

Lemma 4.12. *Let $(D = (V, A), \mathcal{F})$ be a digraft, and suppose $\delta^+(U)$ is a contractible dicut. Let $(D_i = (V_i, A_i), \mathcal{F}_i), i = 1, 2$ be the $(U, V \setminus U)$-contractions of $(D, \mathcal{F})$. Then the following statements hold:*

1. **Decomposition:** *For $i \in \{1, 2\}$, $(D_i, \mathcal{F}_i)$ is a digraft; furthermore, if $J \subseteq A$ satisfies $\mathbf{1}_J \in F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\}$, then $J_i := J \cap A_i$ satisfies $\mathbf{1}_{J_i} \in F(D_i, \mathcal{F}_i)$.*

2. **Composition:** *If $J_i \subseteq A_i$ satisfies $\mathbf{1}_{J_i} \in F(D_i, \mathcal{F}_i), i = 1, 2$, and $J_1 \cap \delta^+(U) = J_2 \cap \delta^+(U)$, then $J := J_1 \cup J_2$ satisfies $\mathbf{1}_J \in F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\}$.*

Proof. **(1)** It can be readily checked that $(D_i, \mathcal{F}_i)$ is a digraft, for each $i \in \{1, 2\}$. Suppose $J \subseteq A$ satisfies $\mathbf{1}_J \in F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\}$. Observe that $F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\} = F(D, \mathcal{F} \cup \{U\})$. Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F} \cup \{U\}$ for $(D, \mathcal{F} \cup \{U\})$. Then, by definition, $\mathbf{1}_J \in F(D, \overline{\mathcal{F}})$. It can be readily checked now that $J_i := J \cap A_i$ satisfies $\mathbf{1}_{J_i} \in F(D_i, \mathcal{F}_i)$, for each $i \in \{1, 2\}$.

(2) We follow the notation in Definition 4.11. Suppose $J_1 \cap \delta^+_D(U) = J_2 \cap \delta^+_D(U) = \{a\}$. Then $J \cap \delta^+_D(U) = \{a\}$. Let $x^i := \mathbf{1}_{J_i} \in F(D_i, \mathcal{F}_i), i = 1, 2$, and let $x := \mathbf{1}_J \in \mathbb{R}^A$. Clearly, $x(\delta^+(U)) = 1$. Let $C := \delta^+_D(W)$ be a dicut of D. We need to show that $x(C) \geq 1$, and equality holds if $W \in \mathcal{F}$.

If $W \subseteq U_1$ or $W \supseteq U_2$, then C is also a dicut of D_2, and $J \cap C = J_2 \cap C \neq \emptyset$, so $x(C) = x^2(C) \geq 1$. Furthermore, if $W \in \mathcal{F}$ and $W \subseteq U_1$, then $W \in \mathcal{F}_2$ so $x(C) = x^2(C) = 1$, and if $W \in \mathcal{F}$ and $W \supseteq U_2$, then $(W \setminus U_2) \cup \{u_2\} \in \mathcal{F}_2$ so $x(C) = x^2(C) = 1$.

Similarly, if $W \subseteq U_2$ or $W \supseteq U_1$, then C is also a dicut of D_1, $x(C) = x^1(C) \geq 1$, and if $W \in \mathcal{F}$, then $x(C) = 1$.

Otherwise, W and U cross. Subsequently, $C_1 := \delta^+_D(U \cup W)$ is a dicut of both D and D_1, $C_2 := \delta^+_D(U \cap W)$ is a dicut of both D and D_2, and therefore the inequality below holds:

$$x(C) = x(C_1) + x(C_2) - x(\delta^+_D(U)) = x^1(C_1) + x^2(C_2) - 1 \geq 1 + 1 - 1 = 1.$$

Suppose $W \in \mathcal{F}$. It remains to show that $x(C) = 1$. Recall that $\overline{\mathcal{F}}$ is the closure of $\mathcal{F} \cup \{U\}$ for $(D, \mathcal{F} \cup \{U\})$. By Lemma 4.9, $\overline{\mathcal{F}}$ is a crossing family. Thus, given that $W, U \in \mathcal{F} \cup \{U\} \subseteq \overline{\mathcal{F}}$ and U, W cross, it follows that $U \cup W, U \cap W \in \overline{\mathcal{F}}$. Subsequently, by Definition 4.11, $((U \cup W) \setminus U_1) \cup \{u_1\} \in \mathcal{F}_1$ and $U \cap W \in \mathcal{F}_2$, so $x^1(C_1) = x^2(C_2) = 1$, implying in turn that $x(C) = 1$, as required. □

We only use Lemma 4.12 (decomposition) in this section, as composition will not be needed until the next section.

4.3 Nontrivial facet-defining inequalities

As we saw in Lemma 4.12, a contractible dicut (if any) can be used to decompose the digraft into two smaller ones, and under certain conditions we can also compose two solutions to go back. As every non-trivial tight dicut is clearly contractible, we shall repeatedly contract them to decompose our digraft into basic pieces.

Observe that if $(D, \mathcal{F})$ is basic, then every tight dicut is trivial, so $s(D, \mathcal{F}) = 0$ by Remark 4.5.

Given a basic digraft, a key idea that allows for the proof of Theorem 1.9 is to contract more dicuts, namely those that correspond to 'non-trivial' facet-defining inequalities. What enables these contractions is the following crucial theorem, proving that both pieces of the contraction also have zero slack.

Theorem 4.13. *Let $(D = (V, A), \mathcal{F})$ be a basic digraft. Suppose $x(\delta^+(U)) \geq 1$ is a facet-defining dicut inequality for $F(D, \mathcal{F})$ that is not equivalent to $x(\delta^+(u)) \geq 1$ for any active source u. Let $U_1 := U$ and $U_2 := V \setminus U$. Let $(D_i = (V_i, A_i), \mathcal{F}_i), i = 1, 2$ be the $(U, V \setminus U)$-contractions of $(D, \mathcal{F})$. Then the following statements hold for each $i \in \{1, 2\}$:*

1. *U_i contains an active source of $(D, \mathcal{F})$,*
2. *every active source for $(D, \mathcal{F})$ in U_{3-i} is an active source for $(D_i, \mathcal{F}_i)$, and vice versa,*
3. *$s(D_i, \mathcal{F}_i) = 0$.*

Proof. Let $U_1 := U$, $U_2 := V \setminus U$, and write $V_i = \{u_i\} \cup U_{3-i}$ for $i = 1, 2$.

Claim 1. *For each $i \in \{1, 2\}$, U_i contains an active source of $(D, \mathcal{F})$, say z_i. Thus, part* **(1)** *holds.*

Proof of Claim. Let $x \in F(D, \mathcal{F})$. Then we have

$$\begin{aligned} x(\delta^+(U)) &= \sum_{u \in \text{sources}(U_1)} x(\delta^+(u)) - \sum_{u \in \text{sinks}(U_1)} x(\delta^-(u)) \\ &= \sum_{v \in \text{sinks}(U_2)} x(\delta^-(v)) - \sum_{v \in \text{sources}(U_2)} x(\delta^+(v)). \end{aligned}$$

Suppose for a contradiction that one of U_1, U_2 does not contain an active source. If U_1 consists solely of tight nodes, then $x(\delta^+(U)) = -\operatorname{disc}(U_1)$, and otherwise, U_2 consists solely of tights nodes, so $x(\delta^+(U)) = \operatorname{disc}(U_2)$. In both cases, we obtain that $x(\delta^+(U))$ has a fixed value for all $x \in F(D, \mathcal{F})$, which is a contradiction as $x(\delta^+(U)) = 1$ determines a facet, hence proper face of $F(D, \mathcal{F})$. ♢

Let V^t, V^a be the sets of tight and active nodes of $(D, \mathcal{F})$, respectively.

Claim 2. *For each $i \in \{1, 2\}$, the set of tight nodes for $(D_i, \mathcal{F}_i)$ is precisely $(V^t \cap U_{3-i}) \cup \{u_i\}$. Thus, part* **(2)** *holds.*

Proof of Claim. As $(D, \mathcal{F})$ is a basic digraft, $\{u\} \in \mathcal{F}$ for each source u in V^t, and $V \setminus u \in \mathcal{F}$ for each sink u in V^t. Thus, by the definition of $(D_i, \mathcal{F}_i)$, the set of its tight nodes contains $(V^t \cap U_{3-i}) \cup \{u_i\}$. To see the reverse inclusion, let $u \in V_i$ be a tight node of $(D_i, \mathcal{F}_i)$. If $u = u_i$, then clearly $u \in (V^t \cap U_{3-i}) \cup \{u_i\}$. Otherwise, $u \in U_{3-i}$.

We first prove that if $x \in F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\}$ then $x(\delta(u)) = 1$. It suffices to show this for $0, 1$ vectors x. To this end, let $J \subseteq A$ such that $\mathbf{1}_J \in F(D, \mathcal{F}) \cap \{x : x(\delta^+(U)) = 1\}$. Then by Lemma 4.12 (decomposition), $\mathbf{1}_{J_i} \in F(D_i, \mathcal{F}_i)$ for $J_i = J \cap A_i$, so given that u is a tight node for $(D_i, \mathcal{F}_i)$, we obtain that $|J_i \cap \delta_{D_i}(u)| = 1$, which implies that $|J \cap \delta_D(u)| = 1$.

Subsequently, the facet of $F(D, \mathcal{F})$ obtained by setting $x(\delta^+(U)) \geq 1$ to equality satisfies the equation $x(\delta(u)) = 1$. Given that $x(\delta^+(U)) \geq 1$ is a facet-defining inequality for $F(D, \mathcal{F})$ that is not equivalent to $x(\delta(v)) \geq 1$ for any active source v, it follows that u must be a tight node of $(D, \mathcal{F})$. Thus, $u \in V^t \cap U_{3-i}$. ♢

It remains to prove part **(3)**, i.e., $s(D_i, \mathcal{F}_i) = 0, i = 1, 2$. Suppose for a contradiction that $s(D_i, \mathcal{F}_i) \geq 1$ for some $i \in \{1, 2\}$. By Theorem 4.7, $(D_i, \mathcal{F}_i)$ has a non-trivial tight dicut $\delta_{D_i}(W)$ such that W separates a pair of active sources of $(D_i, \mathcal{F}_i)$, say $u \in V_i \setminus W$ and $w \in W$, which are also active for $(D, \mathcal{F})$ by Claim 2. Note that $u, w \neq u_i$.

We may assume that $u_i \notin W$ by replacing W with $V_i \setminus W$, if necessary. Let us now consider the dicut $\delta_D(W)$ in D. (Note that $\delta_D(W)$ is equal to one of $\delta_D^{\pm}(W)$.) Note that $W \cap U_i = \emptyset$. Furthermore, each of $W, U_i, V \setminus (W \cup U_i)$ contains an active source of $(D, \mathcal{F})$, namely $w \in W, z_i \in U_i$, and $u \in V \setminus (W \cup U_i)$, where z_i comes from Claim 1.

Claim 3. $F(D, \mathcal{F}) \cap \{x : x(\delta_D(U_i)) = 1\} \subseteq \{x : x(\delta_D(W)) = 1\}$.

Proof of Claim. It suffices to prove this inclusion for $0, 1$ vectors x. To this end, take $J \subseteq A$ such that $\mathbf{1}_J \in F(D, \mathcal{F}) \cap \{x : x(\delta_D^+(U)) = 1\}$. Let $J_i := J \cap A_i$. Then $\mathbf{1}_{J_i} \in F(D_i, \mathcal{F}_i)$ by Lemma 4.12 (decomposition). Thus, $|J_i \cap \delta_{D_i}(W)| = 1$ as $\delta_{D_i}(W)$ is a tight dicut for $(D_i, \mathcal{F}_i)$, implying in turn that $|J \cap \delta_D(W)| = 1$, as required. $\diamondsuit$

It follows from Claim 3 that $\delta_D(W)$ is either a tight dicut for $(D, \mathcal{F})$, or $x(\delta_D(W)) \geq 1$ is a dicut inequality that defines the same facet of $F(D, \mathcal{F})$ as $x(\delta_D(U_i)) \geq 1$. The former is not possible as $(D, \mathcal{F})$ is a basic digraft and $|W| \neq 1, |V| - 1$. Subsequently, $x(\delta_D(W)) = 1$ and $x(\delta_D(U_i)) = 1$ must define the same facet. This implies that the first equation must be implied by the second equation, together with all the equations that define the affine hull of $F(D, \mathcal{F})$, which are of the form $x(\delta(v)) = 1,\ v \in V^t$ because $(D, \mathcal{F})$ is basic.

Subsequently, we must have that $\mathbf{1}_{\delta(W)}$ is in the linear hull of $\mathbf{1}_{\delta(U_i)}$ and $\mathbf{1}_{\delta(v)}, v \in V^t$. Subsequently, the vectors $\mathbf{1}_{\delta(W)}, \mathbf{1}_{\delta(U_i)}$ and $\mathbf{1}_{\delta(v)}, v \in V^t$ are linearly dependent. We have

$$\pm\mathbf{1}_{\delta(W)} = \sum_{v \in \text{sources}(W)} \mathbf{1}_{\delta(v)} - \sum_{v \in \text{sinks}(W)} \mathbf{1}_{\delta(v)}$$
$$\pm\mathbf{1}_{\delta(U_i)} = \sum_{v \in \text{sources}(U_i)} \mathbf{1}_{\delta(v)} - \sum_{v \in \text{sinks}(U_i)} \mathbf{1}_{\delta(v)}.$$

Thus, after applying elementary 'row' operations, the following vectors are linearly dependent:

$$\sum_{v \in \text{sources}(W) \cap V^a} \mathbf{1}_{\delta(v)}$$
$$\sum_{v \in \text{sources}(U_i) \cap V^a} \mathbf{1}_{\delta(v)}$$
$$\mathbf{1}_{\delta(v)} \quad \forall v \in V^t.$$

Note that $\text{sources}(W) \cap V^a \ni w$ and $\text{sources}(U_i) \cap V^a \ni z_i$ are nonempty and disjoint. Thus, given that $u \in (V \setminus W \setminus U_i) \cap V^a$, the linear dependence of the vectors above implies that of the vectors $\mathbf{1}_{\delta(v)}, v \in V \setminus u$, which is a contradiction to Remark 4.2. This finishes the proof of (3). $\square$

4.4 Proof of the Affine Critical Lemma

Recall that by definition, a digraft $(D, \mathcal{F})$ is affine critical if $\text{aff}(F(D, \mathcal{F})) = \{x : x(\delta(v)) = 1, \forall v \in V^t\}$. As an immediate consequence of Remark 4.5, we obtain the following remark.

Remark 4.14. *A digraft is affine critical if and only if it has slack zero.*

While every basic digraft is affine critical, the converse may not necessarily hold, as having slack zero is not sufficient for being basic. We are now ready to prove the Affine Critical Lemma.

Proof of Lemma 1.16. Let $(D = (V, A), \mathcal{F})$ be a basic digraft that is not robust, that is, there is a facet-defining dicut inequality $x(\delta^+(U)) \geq 1$ for $F(D, \mathcal{F})$ that is not equivalent to $x(\delta^+(u)) \geq 1$ for any active source u. Clearly, $s(D, \mathcal{F}) = 0$. By Theorem 4.13, each of $(D, \mathcal{F}), (D_i, \mathcal{F}_i), i = 1, 2$ contains an active source, for $i \in \{1, 2\}$ every active source for $(D, \mathcal{F})$ in U_{3-i} is an active source for $(D_i, \mathcal{F}_i)$ and vice versa, and $s(D_i, \mathcal{F}_i) = 0, i = 1, 2$, so by Remark 4.14, $(D_i, \mathcal{F}_i), i = 1, 2$ are affine critical digrafts, thus finishing the proof. □

5 Proof of Theorem 1.9

The proof proceeds by induction, with the base case being basic robust digrafts.

Proof of Theorem 1.12. Let $(D = (V, A), \mathcal{F})$ be a basic robust digraft. We need to show that $F(D, \mathcal{F})$ has the integer decomposition property, and $\mathrm{aff}(F(D, \mathcal{F})) = \{x : Mx = \mathbf{1}\}$ for some $M \in \mathbb{Z}^{m \times n}$ with $m \geq 1$. To this end, let V^t, V^a be the sets of tight and active nodes of $(D, \mathcal{F})$, respectively. Let $P := F(D, \mathcal{F})$. It follows from the hypothesis that every facet-defining inequality of P is either equivalent to $x_a \geq 0$ for some $a \in A$, or $x(\delta^+(v)) \geq 1$ for some $v \in V^a$. Furthermore, by Corollary 4.1, $\mathrm{aff}(P)$ is described by $x(\delta(v)) = 1$, $\forall v \in V^t$; this proves the second part of the theorem. For the first part, let us write

$$P = \left\{x \in \mathbb{R}^A : x \geq \mathbf{0};\ x(\delta^+(v)) \geq 1,\ \forall v \in V^a;\ x(\delta(v)) = 1,\ \forall v \in V^t\right\}.$$

Take an integer $k \geq 1$, and let $x \in \mathbb{Z}^A_{\geq 0}$ be an integral vector in kP. That is, $x \geq \mathbf{0}$, $x(\delta(v)) = k$ for all $v \in V^t$, and $x(\delta^+(v)) \geq k$ for all $v \in V^a$. By a result of de Werra ([7], see [16], Corollary 1.4.21) on edge-colourings of bipartite graphs, we can write x as the sum of $x^1, \ldots, x^k \in \mathbb{Z}^A_{\geq 0}$ such that for each i, $x^i(\delta(v)) = 1$ for all $v \in V^t$ and $x^i(\delta^+(v)) = \lfloor x(\delta^+(v))/k \rfloor$ or $\lceil x(\delta^+(v))/k \rceil$ for all $v \in V^a$. In particular, each x^i belongs to P. This finishes the proof. □

For the induction step, we will need to compose integral bases along contractible dicuts. The following technical though straightforward lemma will be useful for this purpose. Given subsets $A_1, A_2 \subseteq A$ such that $A_1 \cap A_2 = C$ and $A_1 \cup A_2 = A$, and weights $w^i \in \mathbb{R}^{A_i}$ such that $w^1_a = w^2_a$, $\forall a \in C$, we define $z := w^1 \odot w^2 \in \mathbb{R}^A$ as follows: $z_a := w^i_a$ if $a \in A_i \setminus A_{3-i}$, and $z_a := w^1_a = w^2_a$ if $a \in C$.

Lemma 5.1. *Let $(D = (V, A), \mathcal{F})$ be a digraft, and suppose $\delta^+(U)$ is a contractible dicut. Let $(D_i = (V_i, A_i), \mathcal{F}_i), i = 1, 2$ be the $(U, V \setminus U)$-contractions of $(D, \mathcal{F})$. Suppose*

$$B_1 := \{x^1, \ldots, x^{d_1}\} \subseteq F(D_1, \mathcal{F}_1) \cap \{0, 1\}^{A_1}$$
$$B_2 := \{y^1, \ldots, y^{d_2}\} \subseteq F(D_2, \mathcal{F}_2) \cap \{0, 1\}^{A_2}$$

are integral bases for $\mathrm{lin}(F(D_1, \mathcal{F}_1))$ and $\mathrm{lin}(F(D_2, \mathcal{F}_2))$, respectively. For each $a \in \delta^+(U)$, let $I_a := \{i : x^i_a = 1\}$ and $J_a := \{j : y^j_a = 1\}$. Then both I_a, J_a are nonempty. Furthermore, write $I_a = \{i_1, \ldots, i_k\}$ and $J_a = \{j_1, \ldots, j_\ell\}$, and let

$$z^a_t := x^{i_1} \odot y^{j_t} \qquad t = 1, \ldots, \ell$$
$$z^a_{\ell+t} := x^{i_{1+t}} \odot y^{j_1} \qquad t = 1, \ldots, k - 1.$$

Let $B_1 \odot B_2 := \{z^a_i : a \in \delta^+(U), 1 \leq i \leq |I_a| + |J_a| - 1\}$. Then the following statements hold:

1. *$B_1 \odot B_2 \subseteq F(D,\mathcal{F}) \cap \{0,1\}^A \cap \{x : x(\delta^+(U)) = 1\}$, $|B_1 \odot B_2| = d_1 + d_2 - |\delta^+(U)|$, and $B_1 \odot B_2$ is linearly independent,*

2. *if x is an integer linear combination of the vectors in B_1 and y of B_2, where $x_a = y_a \, \forall a \in \delta^+(U)$, then $x \odot y$ is an integer linear combination of the vectors in $B_1 \odot B_2$,*

3. *$B_1 \odot B_2$ is an integral basis for its linear hull,*

4. *if $\delta^+(U)$ is a tight dicut, then $\operatorname{lin}(B_1 \odot B_2) = \operatorname{lin}(F(D,\mathcal{F}))$.*

Proof. Given that $\delta^+(U)$ is a contractible dicut, it follows that $(D_i, \mathcal{F}_i), i = 1, 2$ are digrafts, so by Corollary 4.1, both I_a, J_a are nonempty. Let $B := B_1 \odot B_2$.

(1) It follows from Lemma 4.12 (composition) that $B \subseteq F(D,\mathcal{F}) \cap \{0,1\}^A$. Furthermore, it is clear from construction that $x(\delta^+(U)) = 1$ for all $x \in B$, and $|B| = d_1 + d_2 - |\delta^+(U)|$. To prove linear independence, suppose $\sum_{a,i} \lambda_i^a z_i^a = 0$ for some $\lambda_i^a \in \mathbb{R}$ for all $a \in \delta^+(U), 1 \le i \le |I_a| + |J_a| - 1$. Fix $a \in \delta^+(U)$ with $I_a = \{i_1, \ldots, i_k\}$ and $J_a = \{j_1, \ldots, j_\ell\}$. Given that B_1 is linearly independent, then for each x^{i_t}, the sum of the coefficients of vectors in B of the form $x^{i_t} \odot y$ for some y, must be 0. Subsequently, we have

$$\sum_{i=1}^{\ell} \lambda_i^a = 0 \tag{4}$$

$$\lambda_{\ell+1}^a = \cdots = \lambda_{\ell+k-1}^a = 0 \tag{5}$$

where (4) computes the coefficient for $x^{i_1} \odot y$, while (5) computes the coefficients for $x^{i_t} \odot y, t = 2, \ldots, k$. Similarly, given that B_2 is linearly independent, for each y^{j_t}, the sum of the coefficients of vectors in B of the form $x \odot y^{j_t}$ for some x, must be 0. Subsequently, we obtain that

$$\sum_{i=\ell}^{\ell+k-1} \lambda_i^a = 0 \tag{6}$$

$$\lambda_2^a = \cdots = \lambda_\ell^a = 0 \tag{7}$$

where (6) computes the coefficient for $x \odot y^{j_1}$, while (7) computes the coefficients for $x \odot y^{j_t}, t = 2, \ldots, \ell$. Observe that (4) and (7) imply that $\lambda_1^a = 0$, so together with (5), we obtain that $\lambda_i^a = 0$ for all $1 \le i \le k + \ell - 1$. As this holds for all $a \in \delta^+(U)$, we obtain that $\lambda_i^a = 0$ for all $a \in \delta^+(U), 1 \le i \le |I_a| + |J_a| - 1$.

(2) Suppose $x = \sum_i \alpha(x^i) x^i$ and $y = \sum_j \beta(y^j) y^j$ for integers $\alpha(x^i)$ and $\beta(y^j)$. Fix $a \in \delta^+(U)$ with $I_a = \{i_1, \ldots, i_k\}$ and $J_a = \{j_1, \ldots, j_\ell\}$. Now choose λ_i^a for all $1 \le i \le k + \ell - 1$ such that

$$\sum_{i=1}^{\ell} \lambda_i^a = \alpha(x^{i_1}) \tag{8}$$

$$\lambda_{\ell+t-1}^a = \alpha(x^{i_t}) \quad t = 2, \ldots, k \tag{9}$$

$$\lambda_t^a = \beta(y^{j_t}) \quad t = 2, \ldots, \ell. \tag{10}$$

(9) and (10) give us the values for $\lambda_t^a, t = 2, \ldots, \ell + k - 1$, all of which are clearly integral. Furthermore, (8) and (10) give us an integer value for λ_1^a:

$$\lambda_1^a = \alpha(x^{i_1}) - \sum_{t=2}^{\ell} \beta(y^{j_t}).$$

Since $x_a = y_a$, it can be readily checked that $\alpha(x^{i_1}) - \sum_{t=2}^{\ell} \beta(y^{j_t}) = \beta(y^{j_1}) - \sum_{t=2}^{k} \alpha(x^{i_t})$, so

$$\sum_{i=\ell}^{\ell+k-1} \lambda_i^a = \beta(y^{j_1}). \tag{11}$$

It follows from (8)-(11) that $x \odot y = \sum_{a,i} \lambda_i^a z_i^a$, as required.

(3) Let $f \in \mathrm{lin}(B) \cap \mathbb{Z}^A$. Observe that $f = x \odot y$, where $x \in \mathrm{lin}(B_1) \cap \mathbb{Z}^{A_1}$ and $y \in \mathrm{lin}(B_2) \cap \mathbb{Z}^{A_2}$. As B_1 (resp. B_2) is an integral basis, x (resp. y) must be an integer linear combination of the vectors in the set, so by part (2), $f = x \odot y$ is an integer linear combination of the vectors in B.

(4) Clearly, $\mathrm{lin}(B) \subseteq \mathrm{lin}(F(D, \mathcal{F}))$. For the reverse inclusion, pick a $0,1$ vector $f \in F(D, \mathcal{F})$. Take $J \subseteq A$ such that $\mathbf{1}_J = f$, and let $J_i := J \cap A_i, i = 1, 2$. As $\delta^+(U)$ is a tight dicut, we have $f(\delta^+(U)) = 1$, so we get from Lemma 4.12 (decomposition) that $f_i := \mathbf{1}_{J_i} \in F(D_i, \mathcal{F}_i), i = 1, 2$. Subsequently, $f_i \in \mathrm{lin}(B_i), i = 1, 2$, so $f = f_1 \odot f_2 \in \mathrm{lin}(B)$. □

We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9. Let $(D = (V, A), \mathcal{F})$ be a digraft. Our goal is to prove that $F(D, \mathcal{F}) \cap \{0, 1\}^A$ contains an integral basis for $\mathrm{lin}(F(D, \mathcal{F}))$. To this end, we may assume that whenever $\delta^+(U)$ is a tight dicut, then U belongs to $\mathcal{F}$, by adding it to the family if necessary. Note that this operation does not change the face $F(D, \mathcal{F})$.

Base case. We shall proceed by induction. If $(D, \mathcal{F})$ is a basic robust digraft, then $F(D, \mathcal{F})$ has the integer decomposition property, and $\mathrm{aff}(F(D, \mathcal{F})) = \{x : Mx = \mathbf{1}\}$ for some $M \in \mathbb{Z}^{m \times n}$ with $m \geq 1$, by Theorem 1.12. It therefore follows from Theorem 1.13 that $F(D, \mathcal{F}) \cap \{0, 1\}^A$ contains an integral basis B for $\mathrm{lin}(F(D, \mathcal{F}))$, so we have proved the base case. For the induction step, let $(D, \mathcal{F})$ be a digraft that is either non-basic or basic non-robust.

Non-basic case. Assume in the first case that $(D, \mathcal{F})$ is not basic. Thus, by the maximality of $\mathcal{F}$, there exists a tight dicut $\delta^+(U)$ such that $1 < |U| < |V| - 1$. Let $D_i = (V_i, A_i), i = 1, 2$ be the $(U, V \setminus U)$-contractions of $(D, \mathcal{F})$. By the induction hypothesis, $F(D_i, \mathcal{F}_i) \cap \{0, 1\}^{A_i}$ contains an integral basis B_i for $\mathrm{lin}(F(D_i, \mathcal{F}_i))$, for $i \in \{1, 2\}$. Then by Lemma 5.1 parts (1), (3) and (4), $B_1 \odot B_2 \subseteq F(D, \mathcal{F}) \cap \{0, 1\}^A$ is an integral basis for $\mathrm{lin}(F(D, \mathcal{F}))$.

Basic non-robust case. Assume in the remaining case that $(D, \mathcal{F})$ is a basic non-robust digraft. Then there exists a facet-defining dicut inequality $x(\delta^+(U)) \geq 1$ for $F(D, \mathcal{F})$ that is not equivalent to $x(\delta^+(u)) \geq 1$ for any active source u. In particular, $|V| - 1 > |U| > 1$. Let $U_1 := U, U_2 := V \setminus U$, and let $D_i = (V_i, A_i), i = 1, 2$ be the $(U, V \setminus U)$-contractions of $(D, \mathcal{F})$.

By the induction hypothesis, $F(D_i, \mathcal{F}_i) \cap \{0, 1\}^{A_i}$ contains an integral basis B_i for $\mathrm{lin}(F(D_i, \mathcal{F}_i))$, for $i \in \{1, 2\}$. Then by Lemma 5.1 parts (1) and (3), $B' := B_1 \odot B_2 \subseteq F(D, \mathcal{F}) \cap \{0, 1\}^A$ is an integral basis for its linear hull. Unlike the previous case, $\mathrm{lin}(B')$ is no longer the same as $\mathrm{lin}(F(D, \mathcal{F}))$, and we must add at least one element to B'.

By the Jump-Free Lemma (i.e., Lemma 1.17), there exists $b \in F(D, \mathcal{F}) \cap \{0, 1\}^A$ such that $b(\delta^+(U)) = 2$. We claim that $B := B' \cup \{b\} \subseteq F(D, \mathcal{F}) \cap \{0, 1\}^A$ is an integral basis for $\mathrm{lin}(F(D, \mathcal{F}))$.

First, we prove that B is a linear basis for $\mathrm{lin}(F(D,\mathcal{F}))$. Linear independence can be checked easily. To show that B is a linear basis, we count the linear dimension of $F := F(D,\mathcal{F})$, which is $d := 1 + \dim(F)$. We claim that $d = |B|$. To this end, let $F_i := F(D_i,\mathcal{F}_i), i = 1,2$ and $d_i := 1 + \dim(F_i), i = 1,2$. By the Affine Critical Lemma (i.e., Lemma 1.16), $(D,\mathcal{F}),(D_i,\mathcal{F}_i), i = 1,2$ are affine critical digrafts each of which contains an active source. Thus, $\kappa^t(D,\mathcal{F}) = \kappa^t(D_1,\mathcal{F}_1) = \kappa^t(D_2,\mathcal{F}_2) = 0$, and $s(D,\mathcal{F}) = s(D_1,\mathcal{F}_1) = s(D_2,\mathcal{F}_2) = 0$ by Remark 4.14. Subsequently, by the slack formula in Definition 4.4,

$$\begin{aligned} d &= 1 + |A| - |V^t| \\ d_1 &= 1 + |A_1| - |V_1^t| \\ d_2 &= 1 + |A_2| - |V_2^t|, \end{aligned}$$

where $V^t, V_i^t, i = 1,2$ denote the sets of tight nodes of $(D,\mathcal{F}),(D_i,\mathcal{F}_i), i = 1,2$, respectively. By Theorem 4.13 part (2), $V_1^t \cup V_2^t = V^t \cup \{u_1,u_2\}$, implying in turn the first equality below:

$$\begin{aligned} d &= d_1 + d_2 - |\delta^+(U)| + 1 \\ &= |B_1| + |B_2| - |\delta^+(U)| + 1 \\ &= |B'| + 1 \\ &= |B|. \end{aligned}$$

The second and last equalities are clear, while the third equality follows from Lemma 5.1 part (1).

It remains to prove that B is an *integral* basis. To this end, pick an integral vector f in $\mathrm{lin}(F(D,\mathcal{F}))$. We now know that f can be expressed as a linear combination of the vectors in B; let $\lambda_z \in \mathbb{R}$ be the coefficient of $z \in B$. Given that f is integral, $f(\delta^+(U))$ is an integer, which can be calculated alternatively as follows:

$$f(\delta^+(U)) = \sum_{z\in B} \lambda_z z(\delta^+(U)) = 2\lambda_b + \sum_{z\in B'} \lambda_z = \lambda_b + \mathbf{1}^\top \lambda.$$

Here we have used $z(\delta^+(U)) = 1$ for all $z \in B'$, guaranteed by Lemma 5.1 part (1). As $B \subseteq F(D,\mathcal{F})$, we have $z(\delta(v)) = 1, \forall z \in B$ for any fixed tight node v. Subsequently, $\mathbf{1}^\top\lambda = \sum_{z\in B}\lambda_z = f(\delta(v))$ is an integer, implying in turn that $\lambda_b = f(\delta^+(U)) - \mathbf{1}^\top\lambda$ is an integer. Now let $f' := f - \lambda_b b \in \mathbb{Z}^A$. Evidently, $f' \in \mathrm{lin}(B') \cap \mathbb{Z}^A$, so given that B' is an integral basis for its linear hull, f' is an integer linear combination of the vectors in B', implying in turn that λ is an integral vector. Thus, $B \subseteq F(D,\mathcal{F}) \cap \{0,1\}^A$ is an integral basis for $\mathrm{lin}(F(D,\mathcal{F}))$, thereby completing the induction step. □

6 Proofs of Theorem 1.1 and the applications

In this section, we start off by a useful mapping of the strengthening sets of a digraph to dijoins of another digraph. We then prove a general lattice theoretic fact about faces of $\mathrm{SCR}(D)$ for a digraph D. After that, we prove Theorem 1.1 and its three applications, namely, Theorem 1.3, Theorem 1.4, and finally Theorem 1.6. Finally, we provide an important example showing that many of our results are best possible.

6.1 A useful mapping

The following theorem will be particularly useful for two of the proofs. The construction given in the proof has appeared before in the literature, e.g., [20, 5].

Theorem 6.1. *Let $D=(V,A)$ be a digraph whose underlying undirected graph is 2-edge-connected. Let $\mathcal{F}$ be a family over ground set V such that $\emptyset, V \notin \mathcal{F}$, and the following face of $\mathrm{SCR}(D)$ is nonempty:*

$$F := \mathrm{SCR}(D) \cap \left\{x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|, \forall U \in \mathcal{F}\right\}.$$

Then there exists a digraft $(D', \mathcal{F}')$ such that the mapping $x \mapsto \binom{x}{\mathbf{1}-x}$ defines a bijection between the face F of $\mathrm{SCR}(D)$ and the face $F(D',\mathcal{F}')$ of $\mathrm{DIJ}(D')$.

Proof. Let $D'=(V',A')$ be the digraph obtained from D by replacing every arc $a := (r,s) \in A$, by the two arcs $(r,t_a),(s,t_a)$, where t_a is a new vertex. Subsequently, $V' = V \cup \{t_a : a \in A\}$ and $A' = \{(r,t_a),(s,t_a) : a=(r,s) \in A\}$. Note that the nodes of D' in V are sources, while the new nodes in $\{t_a : a \in A\}$ are sinks. Furthermore, the underlying undirected graph of D' is 2-edge-connected, as this is so for D.

Given a nonempty proper subset U of V, denote by

$$\varphi(U) := U \cup \{t_a : a=(r,s) \in A;\, r,s \in U\}.$$

Note that $\delta^+_{D'}(\varphi(U)) = \{(r,t_a) : a=(r,s) \in \delta^+_D(U)\} \cup \{(s,t_a) : a=(r,s) \in \delta^-_D(U)\}$ and $\delta^-_{D'}(\varphi(U)) = \emptyset$. Thus, φ maps every nonempty proper subset of V to a dicut of D'. Conversely, it can be readily checked for $U' \subset V'$ that, if $\delta^+_{D'}(U')$ is a minimal dicut of D' such that $|U'| < |V'|-1$, then $U := U' \cap V$ is a nonempty proper subset of V, and $\delta^+_{D'}(U') = \delta^+_{D'}(\varphi(U))$.

Given $J \subseteq A$, denote by

$$\phi(J) := \{(r,t_a) : a=(r,s) \in J\} \cup \{(s,t_a) : a=(r,s) \in A \setminus J\}.$$

Using the mapping φ defined earlier, it can be readily checked that J is a strengthening set in D if, and only if, $J' := \phi(J)$ is a dijoin of D' such that $|J' \cap \delta(t_a)| = 1$, $\forall a \in A$. Subsequently, ϕ is a bijection between the strengthening sets J in D and the dijoins J' in D' such that $|J' \cap \delta(t_a)| = 1$, $\forall a \in A$.

Let

$$\mathcal{F}' := \{\varphi(U) : U \in \mathcal{F}\} \cup \{V' \setminus t_a : a \in A\},$$

and let $F' := F(D',\mathcal{F}')$. For $x \in \mathbb{R}^A$, define $x' \in \mathbb{R}^{V'}$ as follows: for $a=(r,s) \in A$, let $x'_{(r,t_a)} = x_a$ and $x'_{(s,t_a)} = 1 - x_a$. Then

$$\begin{aligned} x'(\delta^+_{D'}(\varphi(U))) &= x(\delta^+_D(U)) + |\delta^-_D(U)| - x(\delta^-_D(U)) && \forall U \subset V, U \neq \emptyset \\ x'(\delta^-_{D'}(t_a)) &= x_a + (1-x_a) = 1 && \forall a \in A. \end{aligned}$$

Subsequently, if $x \in \mathrm{SCR}(D)$, then $x' \in \mathrm{DIJ}(D')$. Furthermore, $x \in F$ if, and only if, $x' \in F'$. In particular, $F' \neq \emptyset$, and so $(D',\mathcal{F}')$ is the desired digraft. □

6.2 Integer lattices and faces of the strongly connected re-orientations polytope

Let us recall some basic concepts of the theory of integer lattices; for a reference textbook we recommend ([18], Chapter 1). A subset $L \subseteq \mathbb{R}^A$ is a *lattice* if it is the set of integer linear combinations of finitely many vectors. Alternatively, L is a lattice if it forms a subgroup of $\mathbb{R}^A$ under addition that is *discrete*, that is, there exists an $\varepsilon > 0$ such that every pair of distinct vectors in L are at distance $\geq \varepsilon$. Given a finite subset $G \subset \mathbb{R}^A$, the *lattice generated by* G, denoted $\mathrm{lat}(G)$, is the set of all integer linear combinations of the vectors in G. A *lattice basis for* L is a set B of linearly independent vectors that generates the lattice, i.e., $L = \mathrm{lat}(B)$. A nontrivial fact is that a lattice basis always exists.

Suppose now L is an *integer lattice*, that is, L is a lattice and $L \subseteq \mathbb{Z}^A$. Let $\overline{L} := \mathrm{lin}(L) \cap \mathbb{Z}^A$ which is another integer lattice that contains L. Note that $\overline{L}$ is the 'densest' integer lattice in $\mathrm{lin}(L)$. It is known that $\overline{L}$ can be partitioned into a finite number of lattices, each of which is an integral shift of L, i.e., of the form $L + w := \{v + w : v \in L\}$ for some $w \in \mathrm{lin}(L) \cap \mathbb{Z}^A$. We refer to the number of parts in this partition as the *index of* L and denote it by $\mathrm{ind}(L) \in \mathbb{Z}_{\geq 1}$. Thus, the smaller the index of L, the denser the lattice is. Of particular interest is the case when L is densest possible. Observe that L has index 1 if, and only if, L contains an integral basis for $\mathrm{lin}(L)$.

Theorem 6.2. *Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. Let $\mathcal{F}$ be a family over ground set V such that $\emptyset, V \notin \mathcal{F}$, $1 - |\delta^-(U)| \neq 0$ for some $U \in \mathcal{F}$, and the following face of $\mathrm{SCR}(D)$ is nonempty:*

$$F := \mathrm{SCR}(D) \cap \left\{x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|,\ \forall U \in \mathcal{F}\right\}.$$

Then the following statements hold:

1. *The lattice generated by $F \cap \{0,1\}^A$ has a lattice basis contained in $F \cap \{0,1\}^A$.*
2. *Let $g := \gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\}$. Then $gx \in \mathrm{lat}\left(F \cap \{0,1\}^A\right)$ for all $x \in \mathrm{lin}(F) \cap \mathbb{Z}^A$.*

Proof. Let L be the lattice generated by $F \cap \{0,1\}^A$, $\overline{L} := \mathrm{lin}(F) \cap \mathbb{Z}^A$, and $g := \gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\}$. By Theorem 6.1, there exists a digraft $(D' = (V', A'), \mathcal{F}')$ such that for $F' := F(D', \mathcal{F}')$, the mapping $f : F \to F'$ defined as $f(x) = \binom{x}{\mathbf{1}-x}$ is a bijection. Let L' be the lattice generated by $F(D', \mathcal{F}') \cap \{0,1\}^{A'}$. By Theorem 1.9, there is an integral basis $B' \subseteq F' \cap \{0,1\}^{A'}$ for $\mathrm{lin}(F')$.

Claim. *Let w be an integral vector in $\mathrm{lin}(F)$, expressed as $w = \sum_{x \in F} \lambda_x x$. Then $\mathbf{1}^\top \lambda$ is $\frac{1}{g}$-integral. Furthermore, if $w = \mathbf{0}$, then $\mathbf{1}^\top \lambda = 0$.*

Proof of Claim. Let $\tau := \sum_{x \in F} \lambda_x$. Note that

$$w(\delta^+(U)) - w(\delta^-(U)) = \sum_{x \in F} \lambda_x (1 - |\delta^-(U)|) = \tau(1 - |\delta^-(U)|) \quad \forall U \in \mathcal{F}.$$

As w is integral, we have $\tau(1 - |\delta^-(U)|) \in \mathbb{Z}$ for all $U \in \mathcal{F}$, and so since $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = g$, it follows that τ is $\frac{1}{g}$-integral. Furthermore, if $w = \mathbf{0}$, then as $1 - |\delta^-(U)| \neq 0$ for some $U \in \mathcal{F}$, we have $0 = w(\delta^+(U)) - w(\delta^-(U)) = \tau(1 - |\delta^-(U)|)$, implying in turn that $\tau = 0$. $\diamondsuit$

Let B be the pre-image of B' under f. Observe that $B \subseteq F \cap \{0,1\}^A$. We shall prove that (a) B is linearly independent, (b) B is a lattice basis for L, and (c) $gw \in L$ for all $w \in \mathrm{lin}(F) \cap \mathbb{Z}^A$.

(a) Suppose $\sum_{b \in B} \lambda_b b = \mathbf{0}$. It follows from the claim above that $\mathbf{1}^\top \lambda = 0$, so $\sum_{b \in B} \lambda_b f(b) = \mathbf{0}$. The linear independence of $B' = \{f(b) : b \in B\}$, along with the bijectivity of f, implies that $\lambda = \mathbf{0}$.

(b) By (a), it suffices to show that $\mathrm{lat}(B) = L$. Clearly, $\mathrm{lat}(B) \subseteq L$. For the reverse inclusion, let $w \in L$. Then $w = \sum_{x \in F \cap \{0,1\}^A} \lambda_x x$ for some integers $\lambda_x, x \in F \cap \{0,1\}^A$. Let $w' := \sum_{x \in F \cap \{0,1\}^A} \lambda_x f(x)$. As λ is integral, $w' \in L'$, so $w' = \sum_{b \in B} \alpha_b f(b)$ for some integers $\alpha_b, b \in B$. Restricting to the coordinates in A, we obtain that $w = \sum_{b \in B} \alpha_b b \in \mathrm{lat}(B)$.

(c) Let $w \in \operatorname{lin}(F) \cap \mathbb{Z}^A$. Write $w = \sum_{x \in F \cap \{0,1\}^A} \lambda_x x$, and let $\tau := \mathbf{1}^\top \lambda$ which is $\frac{1}{g}$-integral by the claim above. Let $w' := \sum_{x \in F \cap \{0,1\}^A} \lambda_x f(x)$, which is $\frac{1}{g}$-integral as $\tau \in \frac{1}{g}\mathbb{Z}$. Subsequently, gw' is an integral vector in $\operatorname{lin}(F')$, so $gw' = \sum_{b \in B} \alpha_b f(b)$ for some integers $\alpha_b, b \in B$, as B' is an integral basis for $\operatorname{lin}(F')$. Restricting to the coordinates in A, we obtain that $gw = \sum_{b \in B} \alpha_b b \in L$, as promised.

Observe that (b) proves part **(1)**, and (c) proves part **(2)** of the theorem. □

Let us point out a subtle detail about part (1) of Theorem 6.2. A set G of generators may not necessarily contain a lattice basis for $\operatorname{lat}(G)$. For instance, $\operatorname{lat}(\{2,3\}) = \mathbb{Z}$, yet $\{2,3\}$ does not contain a lattice basis for $\mathbb{Z}$. Thus, the claim that $F \cap \{0,1\}^A$ contains a lattice basis is non-trivial.

Let's look at part (2) of Theorem 6.2. This part equivalently states that, for $L := \operatorname{lat}\left(F \cap \{0,1\}^A\right)$ and $\overline{L} := \operatorname{lin}(F) \cap \mathbb{Z}^A$, the quotient group $\overline{L}/L$ is an abelian group where the order of every element divides g. Subsequently, every elementary divisor of $\overline{L}/L$ divides g. This implies in turn that $\operatorname{ind}(L)$ is the product of some divisors of g. Furthermore, if g is a prime number, then $\overline{L}/L$ is an elementary p-primary group. For more on concepts relating to group theory, we refer the interested reader to Dummit and Foote's excellent textbook [8], more specifically, Chapter 5, Theorem 5.

6.3 Proof of the main theorem

Proof of Theorem 1.1. Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. Let $\mathcal{F}$ be a nonempty family over ground set V such that $\emptyset, V \notin \mathcal{F}$, and the following face of $\operatorname{SCR}(D)$ is nonempty:

$$F := \operatorname{SCR}(D) \cap \left\{ x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|, \forall U \in \mathcal{F} \right\}.$$

Suppose $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$. It then follows from Theorem 6.2 part (1) that the lattice L generated by $F \cap \{0,1\}^A$ has a lattice basis $B \subseteq F \cap \{0,1\}^A$. Furthermore, it follows from part (2) that $L = \operatorname{lin}(F) \cap \mathbb{Z}^A$, so $\operatorname{ind}(L) = 1$, implying in turn that B is an integral basis for $\operatorname{lin}(F)$ contained in $F \cap \{0,1\}^A$. □

6.4 Subtractive partitioning of strengthening sets

Proof of Theorem 1.3. Let $\tau \geq 2$ be an integer, and let $D = (V, A)$ be a digraph where the minimum size of a dicut is τ. We will prove that there exists an assignment $\lambda_J \in \mathbb{Z}$ to every strengthening set J intersecting every minimum dicut exactly once, such that $\sum_J \lambda_J \mathbf{1}_J = \mathbf{1}$, $\mathbf{1}^\top \lambda = \tau$, and $\left\{\mathbf{1}_J : \lambda_J \neq 0\right\}$ will be an integral basis for its linear hull.

Let $\mathcal{F}$ be the family of sets $U \subset V, U \neq \emptyset$ such that $\delta^-(U) = \emptyset$ and $|\delta^+(U)| = \tau$. Let $F := \operatorname{SCR}(D) \cap \{x : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|, \forall U \in \mathcal{F}\}$. Observe that $F \cap \{0,1\}^A$ corresponds to the strengthening sets of D that intersect every minimum dicut exactly once.

Since every dicut of D (if any) has size at least τ, it follows that $|\delta^+(U)| + (\tau - 1)|\delta^-(U)| \geq \tau$ for all $U \subset V, U \neq \emptyset$, implying in turn that $x^\star := \frac{1}{\tau}\mathbf{1} \in F$.

Since $\mathcal{F} \neq \emptyset$, then $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$, so we may apply Theorem 1.1 to conclude that $F \cap \{0,1\}^A$ contains an integral basis B for $\operatorname{lin}(F)$. This implies that $\mathbf{1} = \tau x^\star \in \tau F$ is an integral linear combination of the vectors in B, say $\sum_{b \in B} \lambda_b \cdot b$. Furthermore, $\mathbf{1}^\top \lambda = \tau$, because $\mathbf{1}^\top \lambda = \sum_{b \in B} \lambda_b \cdot b(\delta^+(U)) = \mathbf{1}(\delta^+(U)) = \tau$ for any given $U \in \mathcal{F}$. Given that B is an integral basis for $\operatorname{lin}(F)$, it follows that $\{b : \lambda_b \neq 0\}$ is also an integral basis for its linear hull, so we are done. □

6.5 Sparse p-adic optimal packings of dijoins

Proof of Theorem 1.4. Let $D = (V, A)$ be a digraph whose underlying undirected graph is connected. Denote by M_0 the matrix whose columns are labeled by A, and whose rows are the indicator vectors of the dijoins of D. Consider the following pair of dual linear programs:

$$\min\{\mathbf{1}^\top x : M_0 x \geq \mathbf{1}, x \geq \mathbf{0}\} \tag{P_0}$$
$$\max\{\mathbf{1}^\top y : M_0^\top y \leq \mathbf{1}, y \geq \mathbf{0}\}. \tag{D_0}$$

Let p be a prime number. Our goal is to exhibit a p-adic optimal solution to (D_0) with at most $2|A|$ nonzero entries.

It is known that the basic optimal solutions of (P_0) are precisely the indicator vectors of the minimum dicuts of D ([17], see [4], §1.3.4). Let $\tau \geq 1$ be the minimum size of a dicut, which is therefore the common optimal value of the primal and dual. If $\tau = 1$, then any vector y that is a standard unit vector is optimal for the dual, and we are clearly done. Otherwise, $\tau \geq 2$. Let $\mathcal{F}$ be the family of sets $U \subset V, U \neq \emptyset$ such that $\delta^-(U) = \emptyset$ and $|\delta^+(U)| = \tau$. By definition, $\mathcal{F} \neq \emptyset$, so $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$.

By complementary slackness for (P_0) and (D_0), there exists a dijoin, and therefore a minimal dijoin J, which intersects every dicut $\delta^+(U), U \in \mathcal{F}$ exactly once. As every minimal dijoin of D is a strengthening set, $\mathbf{1}_J$ belongs to the following face of $\mathrm{SCR}(D)$:

$$F := \mathrm{SCR}(D) \cap \left\{x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|, \ \forall U \in \mathcal{F}\right\}.$$

As $F \neq \emptyset$, and $\gcd\{1-|\delta^-(U)| : U \in \mathcal{F}\} = 1$, we can apply Theorem 1.1 to conclude that $F \cap \{0,1\}^A$ contains an integral basis B for $\mathrm{lin}(F)$.

Let M_1 be the matrix whose rows are the points in $\mathrm{SCR}(D) \cap \{0,1\}^A$. Since every strengthening set is also a dijoin, it follows that M_1 is a row submatrix of M_0. Let M_2 and M_4 be the row submatrices of M_1 corresponding to the vectors in $F \cap \{0,1\}^A$ and B, respectively.

Claim 1. *The system $M_4^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau$ has an integral solution λ.*

Proof of Claim. Since every dicut of D has size at least τ, it follows that $|\delta^+(U)| + (\tau-1)|\delta^-(U)| \geq \tau$ for all $U \subset V, U \neq \emptyset$, implying in turn that $x^\star := \frac{1}{\tau}\mathbf{1} \in F$. Subsequently, $\mathbf{1} = \tau x^\star$ can be expressed as an integer linear combination of the vectors in B, say $\mathbf{1} = \sum_{b \in B} \lambda_b \cdot b$ for $\lambda_b \in \mathbb{Z}, b \in B$. Note that $\frac{1}{\tau}\sum_{b \in B} \lambda_b = 1$ given that $\frac{1}{\tau}\mathbf{1} \in F$. Thus, λ is the desired solution. $\Diamond$

Consider the following pair of dual linear programs:

$$\min\{\mathbf{1}^\top x : M_1 x \geq \mathbf{1}, x \geq \mathbf{0}\} \tag{P_1}$$
$$\max\{\mathbf{1}^\top y : M_1^\top y \leq \mathbf{1}, y \geq \mathbf{0}\}. \tag{D_1}$$

Given that every strengthening set is a dijoin, and every minimal dijoin is a strengthening set, (P_1) is equivalent to (P_0). Thus, (P_1) is integral and the indicator vector of every minimum dicut of D is optimal for it. Subsequently, the optimal value of (D_1) is τ. By complementary slackness for this pair of linear programs, for any optimal solution $\bar{y}$ for (D_1), we have $\bar{y}_z > 0$ only if $z \in F \cap \{0,1\}^A$.

Claim 2. *The system $M_2^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau, y \geq \mathbf{0}$ has a solution $y^\star$ that is strictly positive.*

Proof of Claim. By Theorem 6.1, there exists a digraft $(D' = (V', A'), \mathcal{F}')$ such that for $F' := F(D', \mathcal{F}')$, the mapping $f : F \to F'$ defined as $f(x) = \binom{x}{\mathbf{1}-x}$ is a bijection. Let $c := \tau f(x^\star) = \binom{\mathbf{1}}{(\tau-1)\mathbf{1}} \in \mathbb{R}^{A'}$. Then the minimum c-weight of a dicut of D' is τ, and every $\delta^+_{D'}(U), U \in \mathcal{F}'$ is a minimum c-weight dicut. Furthermore, it can be readily checked that

(i) every arc of A' appears in a minimum c-weight dicut of D',

(ii) if J' is a dijoin of D' which intersects every minimum c-weight dicut of D' exactly once, then $\mathbf{1}_{J'} \in F' \cap \{0,1\}^{A'}$, and

(iii) if $\mathbf{1}_{J'} \in F' \cap \{0,1\}^{A'}$, then J' is a dijoin of D' which intersects every minimum c-weight dicut of D' exactly once.

As (iii) holds, it follows from strict complementarity that there exists a fractional packing $\hat{y}$ of the dijoins of D' of value τ such that $\hat{y}_{J'} > 0$ for every J' such that $\mathbf{1}_{J'} \in F' \cap \{0,1\}^{A'}$. As (ii) holds, it follows from complementary slackness that $\hat{y}_{J'} = 0$ for every dijoin J' such that $\mathbf{1}_{J'} \notin F' \cap \{0,1\}^{A'}$. Finally, as (i) holds, it follows from complementary slackness that every arc of A' has congestion exactly one in the fractional packing $\hat{y}$.

Let $y^\star \in \mathbb{R}^{F\cap\{0,1\}^A}_{\geq 0}$ be defined as follows: for every $z \in F \cap \{0,1\}^A$, let $y^\star_z = \hat{y}_{J'}$ where $\mathbf{1}_{J'} = f(z)$. Since every arc of A', and in particular A, has congestion exactly one in the fractional packing $\hat{y}$, and since $\hat{y}_{J'} = 0$ for every dijoin J' such that $\mathbf{1}_{J'} \notin F' \cap \{0,1\}^{A'}$, it follows that $M_2^\top y^\star = \mathbf{1}$. By construction, $y^\star_z > 0$ for all $z \in F \cap \{0,1\}^A$. Finally, $\mathbf{1}^\top y^\star = \mathbf{1}^\top \hat{y} = \tau$, so $y^\star$ is the desired optimal solution for (D_1). $\Diamond$

Consider now the polyhedron $P = \{y : M_2^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau, y \geq \mathbf{0}\}$. The existence of $y^\star$ from Claim 2 implies that $\mathrm{aff}(P) = \{y : M_2^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau\}$. Furthermore, the existence of λ from Claim 1 implies that $\mathrm{aff}(P)$ contains an integral, hence p-adic point (recall that M_4 is a row submatrix of M_2). Thus, it follows from ([2], Lemma 2.2) that P contains a p-adic point. Our goal is to find a p-adic point in P of *support size* at most $|A| + |B|$.

We know from Claim 1 that the system $M_4^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau$ in $|B|$ variables has an integral, hence p-adic solution. This system, however, may not have a nonnegative p-adic solution. In what follows, we argue that after adding at most $|A|$ rows from M_2 to M_4 we can guarantee a nonnegative solution as well.

Consider an optimal basic feasible solution $(y^\star, x_0^\star, x^\star)$ to the following bounded linear program

$$\max\left\{x_0 : M_2^\top y = \mathbf{1}; \mathbf{1}^\top y = \tau; y_b - x_0 - x_b = 0,\ \forall b \in B; y \geq \mathbf{0}; x_b \geq 0,\ \forall b \in B; x_0 \geq 0\right\}$$

which is in standard equality form. Since $M_2^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau, y > \mathbf{0}$ is feasible by Claim 2, it follows that $x_0^\star > 0$, so $y_b^\star > 0$ for all $b \in B$. As a basic feasible solution, $(y^\star, x_0^\star, x^\star)$ has support size bounded above by the rank of the coefficient matrix for the equality constraints, which is at most $|A| + 1 + |B|$. Denote by M_3 the row submatrix of M_2 corresponding to the nonzero entries of $y^\star$, of which there are at most $|A| + |B|$. Clearly, $M_3^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau$ has a solution where every entry is greater than 0. Moreover, by design, M_3 contains M_4 and at most $|A|$ other rows from M_2. In particular, $M_3^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau$ has an integral solution, too.

In summary, we found a row submatrix M_3 sandwiched between M_2 and M_4, with at most $|A|+|B|$ rows where $M_3^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau, y \geq \mathbf{0}$ is feasible, and the affine hull $\{y : M_3^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau\}$

contains an integral, hence p-adic point. Therefore, by ([2], Lemma 2.2), the system $M_3^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau, y \geq \mathbf{0}$ has a p-adic solution, implying in turn that $M_0^\top y = \mathbf{1}, \mathbf{1}^\top y = \tau, y \geq \mathbf{0}$ has a p-adic solution with support size at most $|A| + |B| \leq 2|A|$, as required. □

6.6 Strongly connected orientations of hypergraphs

Proof of Theorem 1.6. Let $\tau \geq 2$ be an integer, and let $H = (V, \mathcal{E})$ be a τ-uniform hypergraph such that $d_H(X) \geq \tau$ for all $X \subset V, X \neq \emptyset$. Our goal is to find an assignment $\lambda_O \in \mathbb{Z}$ to every strongly connected orientation $O : \mathcal{E} \to V$ such that

$$\sum_{O(E)=v} \lambda_O = 1 \qquad \forall E \in \mathcal{E}, \forall v \in E,$$

and $|\{O : \lambda_O \neq 0\}| \leq (\tau - 1)|\mathcal{E}| + 1$.

To this end, let D be the bipartite digraph on vertex set $V \cup \{t_E : E \in \mathcal{E}\}$ and arc set $\{(v, t_E) : v \in E, E \in \mathcal{E}\}$. In words, we have introduced a sink t_E for every hyperedge $E \in \mathcal{E}$, and added an arc from every node in E to t_E. Thus, every node in V is a source, and every node in $\{t_E : E \in \mathcal{E}\}$ is a sink of degree τ, as H is τ-uniform.

We have that $d_H(X) \geq \tau$ for all $X \subset V, X \neq \emptyset$, which states equivalently that every dicut of D has size at least τ. In particular, as $\tau \geq 2$, the underlying undirected graph of D is 2-edge-connected.

Since the minimum size of a dicut of D is τ, there exists a fractional packing $y^\star$ of dijoins of D of value τ. Since every sink has degree τ, it follows from complementary slackness that if $y^\star_J > 0$, then $|J \cap \delta_D(t_E)| = 1$ for all $E \in \mathcal{E}$. Since every arc belongs to a minimum dicut, it follows that $\sum_J y^\star_J \mathbf{1}_J = \mathbf{1}$.

Let $\mathcal{F} := \{V(D) \setminus t_E : E \in \mathcal{E}\}$. What we argued above implies that whenever $y^\star_J > 0$, then $\mathbf{1}_J \in F := F(D, \mathcal{F})$. In particular, $F \neq \emptyset$, so $(D, \mathcal{F})$ is a digraft. Thus, by Theorem 1.9, $F \cap \{0,1\}^{A(D)}$ contains an integral basis B for $\text{lin}(F)$. As $\mathbf{1} = \sum_J y^\star_J \mathbf{1}_J \in \text{lin}(F)$, it follows that $\mathbf{1} = \sum_{b \in B} \alpha_b b$ for some integers $\alpha_b, b \in B$.

For every orientation $O : \mathcal{E} \to V$ of H, let $J_O := \{(O(E), t_E) : E \in \mathcal{E}\} \subseteq A(D)$. It can be readily checked that if J_O is a dijoin of D, i.e., if $\mathbf{1}_{J_O} \in F$, then O is a strongly connected orientation of H.

Observe that if $\mathbf{1}_J \in F$, then $J = J_O$ for some orientation O of H, which must be strongly connected as argued above. Thus, every point in $F \cap \{0,1\}^{A(D)}$ corresponds to some strongly connected orientation of H. For every $b \in B$, let O_b be the corresponding strongly connected orientation in H, and let $\lambda_{O_b} := \alpha_b$; let $\lambda_O := 0$ for all other strongly connected orientations O of H. The equality $\sum_{b \in B} \alpha_b b_{(v,t_E)} = 1$ for all $(v, t_E) \in A(D)$, implies that

$$\sum_{O(E)=v} \lambda_O = 1 \qquad \forall E \in \mathcal{E}, \forall v \in E.$$

Furthermore, $|\{O : \lambda_O \neq 0\}| \leq |B| \leq |A(D)| - |\mathcal{E}| + 1 = (\tau - 1)|\mathcal{E}| + 1$, as desired. □

6.7 An example

In this subsection, we give a classic example illustrating that three of our theorems are best possible in a certain sense.

Consider the digraph $D = (V, A)$ shown in Figure 1, and let C be the set of solid arcs [21]. Denote by W the set of the six sources and sinks of D. It can be seen that there are exactly four strengthening sets $J \subseteq C$ of D such that $|J \cap \delta(v)| = 1$ for all $v \in W$. That is, the face F of $\mathrm{SCR}(D)$ obtained by enforcing the following equalities has exactly four integral vectors:

$$\begin{aligned} x(\delta^+(v)) - x(\delta^-(v)) &= 1 - |\delta^-(v)| && \forall v \in W, v \text{ is a source} && (12)\\ x(\delta^+(V \setminus v)) - x(\delta^-(V \setminus v)) &= 1 - |\delta^-(V \setminus v)| && \forall v \in W, v \text{ is a sink} && (13)\\ x_a &= 0 && \forall a \in A \setminus C. && (14) \end{aligned}$$

Observe that the greatest common divisor of the right-hand sides is 1. Denote by $L \subseteq \mathbb{Z}^A$ the lattice generated by the four integral vectors in F.

It can be readily checked that while $\mathbf{1}_C \in \mathbb{Z}^A$ is an integral vector in the linear, in fact conic hull of L, it does not belong to L itself, implying that $\mathrm{ind}(L) > 1$, and also $F \cap \{0,1\}^A$ is not an IGSC. Thus, Theorem 1.1 does not extend to all faces of $\mathrm{SCR}(D)$ where the greatest common divisor of the right-hand sides of the tight constraints is 1.

Furthermore, let F' be the face of $\mathrm{SCR}(D)$ obtained by enforcing just (12) and (13). We know from Theorem 1.1 that $F' \cap \{0,1\}^A$ contains an integral basis. However, as $F \cap \{0,1\}^A$ is not an IGSC, it follows that $F' \cap \{0,1\}^A$ is not an IGSC either; this is because F is a face of F' and so $\mathrm{cone}(F)$ is a face of $\mathrm{cone}(F')$, and being an IGSC is closed under taking faces of the conic hull. Thus, Theorem 1.1 cannot be strengthened to conclude that $F \cap \{0,1\}^A$ is an IGSC.

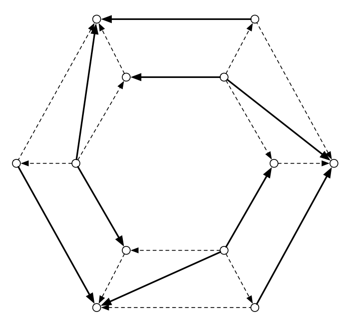

Figure 1: A $0,1$-weighted digraph where the solid and dashed arcs have weights 1 and 0, respectively.

Let us view the arcs inside and outside of C as having capacity 1 and 0, respectively. It can be readily checked that every dicut $\delta^+(U)$ has capacity at least 2, i.e., $|C \cap \delta^+(U)| \geq 2$. Furthermore, every node in W corresponds to a dicut of capacity 2, and these are the only minimum capacity dicuts. Subsequently, the strengthening sets of D contained in C that intersect every minimum dicut exactly once, correspond precisely to the four $0,1$ vectors in $\mathrm{SCR}(D)$ satisfying (12)-(14). However, as we noted before, $\mathbf{1}_C$ cannot be expressed as an integer linear combination of these four $0,1$ vectors, showing that Theorem 1.3 does not extend to the capacitated setting.

Finally, Theorem 1.4 is best possible in the sense that it does not extend to the capacitated setting; let us elaborate. While every dicut of the instance above has capacity at least 2, the set C cannot be decomposed into 2 dijoins [21]. What's more striking about this example is that for any prime number $p \neq 2$, there is no assignment of a p-adic rational number λ_J to every dijoin *contained in* C such that $\mathbf{1}^\top \lambda = 2$, $\sum_J \lambda_J \mathbf{1}_J \leq \mathbf{1}$, and $\lambda \geq 0$. We leave this as an exercise for the reader.

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References

[1] A. Abdi, G. Cornuéjols, B. Guenin, and L. Tunçel. Dyadic linear programming and extensions. *Mathematical Programming*, 2024.

[2] A. Abdi, G. Cornuéjols, B. Guenin, and L. Tunçel. Total dual dyadicness and dyadic generating sets. *Mathematical Programming*, 206(1):125–143, 2024.

[3] A. Abdi, G. Cornuéjols, and M. Zlatin. On packing dijoins in digraphs and weighted digraphs. *SIAM Journal on Discrete Mathematics*, 37(4):2417–2461, 2023.

[4] G. Cornuéjols. *Combinatorial Optimization: Packing and Covering*. Society for Industrial and Applied Mathematics, 2001.

[5] G. Cornuéjols, S. Liu, and R. Ravi. Approximately packing dijoins via nowhere-zero flows. In *Integer Programming and Combinatorial Optimization: 25th International Conference, IPCO 2024, Wroclaw, Poland, July 3–5, 2024, Proceedings*, pages 71–84, Berlin, Heidelberg, 2024. Springer-Verlag.

[6] M. H. de Carvalho, C. L. Lucchesi, and U. S. R. Murty. Optimal ear decompositions of matching covered graphs and bases for the matching lattice. *Journal of Combinatorial Theory, Series B*, 85(1):59–93, 2002.

[7] D. de Werra. Equitable colorations of graphs. *R.A.I.R.O.*, 5(R-3):3–8, 1971.

[8] D. S. Dummit and R. M. Foote. *Abstract Algebra, 3rd Edition*. John Wiley and Sons, Inc., 2004.

[9] J. Edmonds and R. Giles. A min-max relation for submodular functions on graphs. In *Studies in integer programming (Proc. Workshop, Bonn, 1975)*, pages 185–204. Ann. of Discrete Math., Vol. 1, 1977.

[10] J. Edmonds, W. R. Pulleyblank, and L. Lovász. Brick decompositions and the matching rank of graphs. *Combinatorica*, 2(3):247–274, 1982.

[11] A. Frank. *Connections in Combinatorial Optimization*. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, 2011.

[12] S. Fujishige. Structures of polyhedra determined by submodular functions on crossing families. *Math. Programming*, 29:125–141, 1984.

[13] A. M. H. Gerards and A. Sebő. Total dual integrality implies local strong unimodularity. *Mathematical Programming*, 38(1):69–73, 1987.

[14] B. Guenin and S. Hwang. Dyadic packing of dijoins. Submitted, 2024.

[15] L. Lovász. Matching structure and the matching lattice. *Journal of Combinatorial Theory, Series B*, 43(2):187–222, 1987.

[16] L. Lovász and M. D. Plummer. *Matching Theory*, volume 367. AMS Chelsea Publishing, 2009.

[17] C. L. Lucchesi and D. H. Younger. A minimax theorem for directed graphs. *J. London Math. Soc. (2)*, 17(3):369–374, 1978.

[18] J. Martinet. *Perfect Lattices in Euclidean Spaces*. Springer-Verlag Berlin Heidelberg, 2003.

[19] K. Murota. *Discrete Convex Analysis*. Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[20] A. Schrijver. Observations on Woodall's conjecture. Accessed online at https://homepages.cwi.nl/ lex/files/woodall.pdf.

[21] A. Schrijver. A counterexample to a conjecture of Edmonds and Giles. *Discrete Math.*, 32(2):213–215, 1980.

[22] A. Schrijver. *Combinatorial Optimization. Polyhedra and Efficiency.* Springer, Berlin, Heidelberg, 2003.

[23] D. Woodall. Menger and König systems. In Y. Alavi and D. Lick, editors, *Theory and Applications of Graphs.*, volume 642 of *Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg, 1978.