# Integral bases, perfect matchings, and the Petersen graph

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*Dedicated to the memory of Professor Murty*

**Abstract**

Let $G = (V, E)$ be a matching-covered graph, denote by $P$ its perfect matching polytope, and by $L$ the integer lattice generated by the integral points in $P$. In this paper, we give polyhedral proofs for two difficult results established by Lovász (1987), and by Carvalho, Lucchesi, and Murty (2002) in a series of three papers. More specifically, we reprove that $L$ has a lattice basis consisting solely of incidence vectors of some perfect matchings of $G$, $2x \in L$ for all $x \in \mathrm{lin}(P) \cap \mathbb{Z}^E$, and if $G$ has no Petersen brick then $L = \mathrm{lin}(P) \cap \mathbb{Z}^E$. This is achieved by studying the facial structure of $P$ and its relationship with the lattice $L$. Along the way, we give a new polyhedral characterization of the Petersen graph.

**Keywords:** matching-covered graph, matching lattice, lattice basis, Petersen graph, Birkhoff–von Neumann graph, separating cut.

## 1 Introduction

Let $G = (V, E)$ be a matching-covered graph, that is, a graph where every edge appears in a perfect matching. Denote by $P(G)$ the polytope whose vertices are the incidence vectors of the perfect matchings of $G$, and by $L(G)$ the (integer) lattice generated by the integral points in $P(G)$, i.e., $L(G)$ is the set of all integer linear combinations of $P(G) \cap \{0, 1\}^E$.

A seminal result in combinatorial optimization due to Edmonds and Johnson [EJ73] is that $P(G)$ can be described by non-negativity inequalities: $x_e \geq 0, e \in E$, degree equations: $x(\delta(v)) = 1, v \in V$, and *odd cut* inequalities amounting to $x(\delta(U)) \geq 1$ for all odd-sized subsets $U \subset V$. Here, $x(A)$ denotes $\sum_{a \in A} x_a$. Later Seymour [Sey79] gave a graph-theoretic proof of this result using Tutte's characterization of perfect matchings [Tut47]. In the same paper, through a long and technical argument, Seymour proved that in a bridgeless cubic graph, the all-2s vector belongs to $L(G)$, and in fact the all-1s vector belongs to the lattice if $G$ has no Petersen minor, thus establishing weaker variants of the Berge-Fulkerson conjecture and Tutte's 4-flow conjecture, respectively [Ful71, Tut66].

Naddef [Nad82] and Edmonds, Pulleyblank, and Lovász [EPL82] computed the dimension of $P(G)$ based on structural properties of $G$. These lines of questions led to the development of a fascinating area known as *matching theory* [LP09, LM24]. In a seminal paper of the area, Lovász [Lov87] gave a far-reaching extension of Seymour's result and characterized the vectors in $L(G)$, and proved that $2x \in L$ for all $x \in \mathrm{lin}(P(G)) \cap \mathbb{Z}^E$, and if $G$ has no Petersen brick then $L(G) = \mathrm{lin}(P(G)) \cap \mathbb{Z}^E$. Here, $\mathrm{lin}(S)$ denotes the linear hull of $S$ for $S \subseteq \mathbb{R}^E$. The core of his argument was proving a difficult lemma that characterized the dual lattice for a non-Petersen brick. This lengthy proof drew on key tools from matching theory, including the notions of barrier cuts, 2-separation cuts, and ear decompositions.

In a series of three excellent papers [dCLM02a, dCLM02b, dCLM02c], totaling over 120 pages, Carvalho, Lucchesi, and to Murty proved that $G$ admits an ear decomposition that, vaguely speaking, is sensitive to the numbers of bricks and Petersen bricks, and as a key application, they proved that $L(G)$ has a *lattice basis* $B$ consisting solely of incidence vectors of some perfect matchings of $G$, that is, $B$ is a (linear) basis for $\operatorname{lin}(L(G))$, and every vector in $\operatorname{lin}(L(G)) \cap \mathbb{Z}^E$ is an integer linear combination of $B$. This in turn answered a question raised by Murty ([Mur94], Problem 7.3).

**Main Theorem.** In this paper, we give a polyhedral proof of the theorems mentioned due to Lovász, and Carvalho, Lucchesi, and Murty, namely the following combined theorem. A *Petersen brick* is any graph whose simplification is the Petersen graph.

**Theorem 1.1** ([dCLM02c, Lov87]). *Let $G = (V, E)$ be a matching-covered graph, let $L := L(G)$ and $\bar{L} := \operatorname{lin}(P(G)) \cap \mathbb{Z}^E$. Then $L$ has a lattice basis consisting solely of some perfect matchings of $G$. Furthermore, if $G$ has $p$ Petersen bricks in its tight cut decomposition, then*

$$L = \bar{L} \cap \{x : x(A_i) \equiv 0 \pmod 2,\ \forall i \in [p]\},$$

*where each $A_i$ is the edge set of some 5-cycle of the $i^{th}$ Petersen brick. In particular, if $p = 0$ then $L = \bar{L}$, and if $p \geq 1$ then $2x \in L$ for all $x \in \bar{L}$.*

Above, $[p]$ denotes $\{1, 2, \ldots, p\}$ for $p \geq 1$, and $\emptyset$ for $p = 0$. Our proof mostly circumvents the lengthy graph-theoretic arguments, and in particular has no dependence on matching theoretic notions such as 2-separation cuts, braces, removable edges, and ear decompositions, and minimal dependence on the notions of barriers and bricks. It also eliminates the need to study the dual lattice through matching-integral vectors. Instead, our proof is based on polyhedral, or otherwise polyhedrally-driven graph-theoretic arguments that study the relationship between the lattice and the facial structure of the polytope.

**The Integral Basis Theorem.** A matching-covered graph is *Petersen-free* if it has no Petersen brick in its tight cut decomposition. A key notion needed for our proof is that of an *integral basis* for a rational linear subspace, which is a basis $B$ such that every integral vector in the subspace is an integer linear combination of $B$. The crux of Theorem 1.1 is the following special case.

**Theorem 1.2.** *Let $G = (V, E)$ be a Petersen-free matching-covered graph. Then $\operatorname{lin}(P(G))$ has an integral basis consisting solely of the incidence vectors of some perfect matchings. In particular, $L(G) = \operatorname{lin}(P(G)) \cap \mathbb{Z}^E$.*

Our proof of this theorem is based on two key ingredients.

**The Petersen Graph Lemma.** A graph $G = (V, E)$ is *Birkhoff–von Neumann (BvN)* if $P(G) = \left\{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) = 1,\ \forall v \in V\right\}$ [dCKWL20] (see also [Bal81, dCLM04]). That is, $G$ is *not* BvN if, and only if, $P(G)$ has a facet-defining inequality not exposed by a non-negativity inequality. By the Edmonds-Johnson theorem, such a facet must be exposed by (a special type of) an odd cut, which we refer to as a *separating facet-defining cut* of $G$. It is well-known that every bipartite graph is BvN, though some non-bipartite graphs such as $K_4$ can also be BvN. We shall prove the following lemma.

**Lemma 1.3.** *Let $G = (V, E)$ be a brick, and $d$ the (affine) dimension of $P(G)$. Suppose every face of dimension $d - 2$ is exposed by a non-negativity inequality. Then $|V| \leq 10$. Furthermore, either G is the Petersen graph, or G is BvN, or there exist a perfect matching $M$ and a separating facet-defining cut $C$ such that $|M \cap C| = 3$.*

**The Intersection Theorem.** A *near-brick* is a matching-covered graph with exactly one brick in its tight cut decomposition. We shall need the following theorem.

**Theorem 1.4.** *Let $G = (V, E)$ be a Petersen-free non-BvN near-brick. Then there exists a perfect matching $M$ and a separating facet-defining cut $C$ such that $|M \cap C| = 3$.*

This is a special case of a more general result by Campos and Lucchesi [CL00], claiming in particular that in $G$ as above, for every non-tight separating cut $C$, there is a perfect matching $M$ such that $|M \cap C| = 3$. The Intersection Theorem is a slight variant of the main theorem in the two papers [dCLM02a, dCLM02b], which establishes in an arbitrary matching-covered graph the minimum intersection size between a perfect matching and a separating cut with intersection size at least three.

**The main ideas.** Let $d := \dim(P(G))$. The proof of the Intersection Theorem proceeds by first reducing to bricks, then to the case where every facet of $P(G)$ is *not* exposed by a non-negativity inequality, and finally to the case where every $(d-2)$*-face*, i.e., a face of dimension $d - 2$, *is* exposed by a non-negativity inequality. We then resort to the Petersen Graph Lemma to finish the proof.

The proof of the Integral Basis Theorem proceeds by reducing the problem to bricks. If $G$ is BvN, then $P(G)$ has the 'integer decomposition property', which we use to settle the claim. Otherwise, by the Intersection Theorem, there is a separating facet-defining cut $C$ and a perfect matching $M$ such that $|C \cap M| = 3$. After potentially tweaking $(C, M)$, we then decompose $G$ into two smaller Petersen-free matching-covered graphs $G_1$ and $G_2$, find integral bases $B_1$ and $B_2$ for each and compose them, and then add the incidence vector of $M$ to obtain an integral basis for $G$.

**Outline.** In §2, we give a brief overview of necessary notions and results from matching theory, and in §3, we give a short proof of the Petersen Graph Lemma. In §4, we present further preliminary results, and in §5, we prove the Intersection Theorem. Finally, in §6, we prove the Integral Basis Theorem, and then obtain the Main Theorem as a consequence.

## 2 Ingredients for the Petersen Graph Lemma

Fix a matching-covered graph $G = (V, E)$. For an odd cut $C$ and an edge $e \in E$, denote by $P(G; C)$ and $P_e(G)$ the faces $P(G) \cap \{x : x(C) = 1\}$ and $P(G) \cap \{x : x_e = 0\}$ of $P(G)$, respectively.

**2.1. Separating cuts.** Let $C = \delta(X)$ be an odd cut such that $1 < |X| < |V| - 1$. We say that $C$ is *separating*, or *contractible*, in $G$ if $P(G; C)$ is not contained in $\{x : x_e = 0\}$ for any edge $e \in E$. We refer to $G/X, G/\bar{X}$ as the $C$*-contractions* or *cut-contractions* of $G$. We denote the *contraction vertices* in a cut-contraction $G/X$ by the corresponding lower case letter $x$. Observe that $C$ is separating if, and only if, both $C$-contractions are matching-covered. Observe further that if $\delta(X), \delta(Y)$ are odd cuts of $G$ such that $X \subset Y$, where $\delta(Y)$ is separating in $G$, and $\delta(X)$ is separating in $G/\bar{Y}$, then $\delta(X)$ is separating in $G$.

**2.2. Bricks.** A *tight cut* is an odd cut $C = \delta(X)$ such that $1 < |X| < |V| - 1$ and $P(G; C) = P(G)$. Clearly, a tight cut is separating. A matching-covered graph with no tight cut is a *brick* if it is non-bipartite, and is a *brace* otherwise. Cut-contractions along tight cuts repeatedly give rise to a binary tree rooted at $G$ whose leaves correspond to a *tight cut decomposition* of $G$ into bricks and braces. The matching polytope $P(G)$ is known to have dimension $|E| - |V| + 1 - b(G)$, where $b(G)$ is the number of bricks of $G$ in a (or any) tight cut decomposition [Nad82, EPL82]. In fact, the list of bricks and braces in a tight cut decomposition is unique up to the multiplicity of the edges in each brick and brace [Lov87]. Finally, $b(G) = 0$ if and only if $G$ is bipartite [Nad82].

**2.3. Separating facet-defining cuts.** Let $\delta(X)$ be an odd cut that is facet-defining for $P(G)$. In particular, $1 < |X| < |V| - 1$. Observe that $C$ is not separating if, and only if, $x_e \geq 0$ defines the same facet of $P(G)$. Subsequently, $G$ is not BvN if, and only if, there exists a separating facet-defining cut.

# 3 Proof of the Petersen Graph Lemma

Let $v := |V|$ and $e := |E|$, let $f$ be the number of facets and $t$ the number of $(d-2)$-faces of $P(G)$. As $G$ is a brick, it follows that $d = e - v$. Observe that every facet of the $d$-dimensional polytope $P(G)$ has at least $d$ distinct adjacent facets, and each adjacency defines a distinct $(d-2)$-face. Furthermore, every $(d-2)$-face belongs to exactly 2 facets. Thus, $t \geq \frac{f \cdot d}{2}$. Given that every $(d-2)$-face is exposed by $x_e \geq 0$ for some edge $e \in E$, and each such edge defines at most one $(d-2)$-face, it follows that $e \geq t$. Furthermore, $f \geq d + 1$ as $P(G)$ is $d$-dimensional, so

$$e \geq t \geq \frac{fd}{2} \geq \binom{d+1}{2} = \binom{e-v+1}{2}.$$

Subsequently, $e \geq \binom{e-v+1}{2}$, which can be rewritten as $e + v \geq (e-v)^2$. Given that $G$ is a brick, we may assume every vertex $u$ has degree at least 3 (otherwise, $u$ would have 1 or 2 neighbors, and as $G$ is a brick, $v \in \{2, 4\}$). Subsequently, $2e - 3v \geq 0$. Let us now solve the following convex minimization problem:

$$\min \left\{ (x-y)^2 - x - y : -2x + 3y \leq 0, 10 - y \leq 0 \right\}.$$

We see that at the minimum, $x = 15, y = 10$, and the Lagrange multipliers for the inequalities are $\lambda = 9/2$ and $\mu = 5/2$, respectively, and the optimal value is 0. Subsequently, the inequalities $e + v \geq (e-v)^2$ and $2e - 3v \geq 0$ imply that $v \leq 10$, and if $v = 10$ then $e = 15$.

Suppose $G$ is not BvN, and there do not exist a perfect matching $M$ and a separating facet-defining cut $C$ such that $|M \cap C| = 3$. We shall prove that $G$ is the Petersen graph.

We claim that $v = 10$. If not, then $v \leq 8$. As $G$ is not BvN, $P(G)$ has a facet not exposed by a non-negativity inequality, say it is exposed by an odd cut $C = \delta(X)$. Then $C$ must be separating, so $3 \leq |X| \leq |V \setminus X| \leq v - 3$. Thus, $v \in \{6, 8\}$ and $|X| = 3$. Let $M$ be a perfect matching such that $|M \cap C| > 1$. As $|X| = 3$, it follows that $|M \cap C| = 3$, a contradiction.

Thus, $v = 10$, and so $e = 15$ and $e + v = (e-v)^2$, implying that $e = t = \frac{fd}{2} = \binom{d+1}{2}$, and so $f = d + 1 = 6$. In particular, $G$ is a cubic graph as $2e = 3v$, and no facet is exposed by a non-negativity inequality as $e = t$.

Let $C = \delta(X)$ be one of the 6 facet-defining cuts, which must be separating. As there is no perfect matching intersecting $C$ three times, it follows that $|X| = |V \setminus X| = 5$, and $|C| \geq 5$. As connected subgraphs, $G[X], G[V \setminus X]$ each contains at least 4 edges, so $|C| \in \{5, 7\}$ and both of $G[X], G[V \setminus X]$

are either 5-cycles or 4-paths. In the case of the latter, the $C$-contractions are not matching-covered, which is a contradiction as $C$ is separating. Thus, $|C| = 5$ and both of $G[X], G[V \setminus X]$ are 5-cycles. As there is no perfect matching intersecting $C$ three times, there is no 4-cycle intersecting $C$ twice, implying in turn that $G$ is the Petersen graph, as required. $\square$

# 4 Ingredients for the Intersection Theorem

Let $G = (V, E)$ be a matching-covered graph, and let $d := \dim(P(G))$. Let $C = \delta(X)$ be a separating cut, and let $G_1 := G/X$ and $G_2 := G/\bar{X}$. Suppose

$$B_1 := \{x^1, \ldots, x^{\hat{d}_1}\} \subseteq P(G_1) \cap \{0,1\}^{E(G_1)} \quad \text{and} \quad B_2 := \{y^1, \ldots, y^{\hat{d}_2}\} \subseteq P(G_2) \cap \{0,1\}^{E(G_2)}$$

are bases for $\mathrm{lin}(P(G_1))$ and $\mathrm{lin}(P(G_2))$, respectively. Observe that $|B_i| = 1 + \dim(P(G_i))$ for $i \in [2]$. These notations and objects are fixed throughout this section. We proceed to state several preliminary claims, and present proofs for the non-trivial, non-routine statements.

**4.1. Composition along separating cuts.** Given vectors $x \in \mathbb{R}^{E(G_1)}$ and $y \in \mathbb{R}^{E(G_2)}$ such that $x_e = y_e$ for all $e \in C$, we define $z := x \odot y \in \mathbb{R}^E$ as follows: $z_e := x_e$ if $e \in E(G_1) \setminus E(G_2)$, $z_e := y_e$ if $e \in E(G_2) \setminus E(G_1)$, and $z_e := x_e = y_e$ if $e \in C$. For each $e \in C$, let $I_e := \{i \in [\hat{d}_1] : x^i_e = 1\}$ and $J_e := \{j \in [\hat{d}_2] : y^j_e = 1\}$. Then both $I_e, J_e$ are nonempty as $G_1, G_2$ are matching-covered. Write $I_e = \{i_1, \ldots, i_k\}$ and $J_e = \{j_1, \ldots, j_\ell\}$, and let

$$z^e_t := x^{i_1} \odot y^{j_t}, \quad t = 1, \ldots, \ell \quad \text{and} \quad z^e_{\ell+t} := x^{i_{1+t}} \odot y^{j_1}, \quad t = 1, \ldots, k-1.$$

Let $B_1 \odot B_2 := \{z^e_i : e \in C, 1 \leq i \leq |I_e| + |J_e| - 1\}$. This is also known as the *merger operation* ([LM24], §6.3.1). The following statements can be readily checked for $B := B_1 \odot B_2$ (see §A for a proof).

(1) $B \subseteq P(G; C) \cap \{0,1\}^E$, $|B| = |B_1| + |B_2| - |C|$, and $B$ is a basis for $\mathrm{lin}(P(G; C))$.

(2) If $B_i$ is a lattice basis for $L(G_i)$ for $i \in [2]$, then $B$ is a lattice basis for the lattice generated by the integral points in $P(G; C)$.

(3) If $B_i$ is an integral basis for $\mathrm{lin}(P(G_i))$ for $i \in [2]$, then $B$ is an integral basis for $\mathrm{lin}(P(G; C))$.

Furthermore, we have the following.

(4) Suppose $B_1 \setminus \{x^1\} \subseteq \{x : x(D) = 1\}$ for some $D \subseteq E(G_1)$, and $B_1 \setminus \{x^1\} \not\subseteq \{x : x_f = 0\}$ for any $f \in E(G_1)$. We can then apply the composition procedure in such a way that for some $z^\star \in B$, we have $B \setminus \{z^\star\} \subseteq \{z : z(D) = 1\}$, $B \setminus \{z^\star\} \not\subseteq \{z : z_f = 0\}$ for any $f \in E$, and $z^\star(D) = x^1(D)$.

To achieve this, for the element $e \in C$ such that $x^1_e = 1$, we index $I_e$ such that $x^{i_1} \neq x^1$, which is possible as $B_1 \setminus \{x^1\} \not\subseteq \{x : x_e = 0\}$ and so $k \geq 2$. This guarantees that $x^1$ is composed only once with some other vector $y^j$ in $B_2$. We set $z^\star := x^1 \odot y^j$.

**4.2. Uncrossing odd cuts.** Let $C_1 = \delta(X_1), C_2 = \delta(X_2)$ be separating cuts where $|X_1 \cap X_2|$ is odd, and $X_1, X_2$ *cross*, meaning that $X_1 \cap X_2, X_1 \setminus X_2, X_2 \setminus X_1, \overline{X_1 \cup X_2}$ are nonempty. Suppose further that $P(G; C_1) \cap P(G; C_2) \not\subseteq \{x : x_e = 0\}$ for any edge $e \in E$. In particular, no edge of $G$ goes from $X_1 \setminus X_2$ to $X_2 \setminus X_1$, as such an edge would not belong to any perfect matching intersecting both $C_1, C_2$ exactly once. Let $I := \delta(X_1 \cap X_2)$ and $U := \delta(X_1 \cup X_2)$, both of which are odd cuts of $G$. As there is no edge from $X_1 \setminus X_2$ to $X_2 \setminus X_1$, it follows that $x(C_1) + x(C_2) = x(I) + x(U)$. Thus, $P(G; C_1) \cap P(G; C_2) = P(G; I) \cap P(G; U)$.

**4.3. BvN cut-contractions.** Suppose both $G_1, G_2$ are BvN. Then every facet of the polytope $P(G; C)$ is exposed by a non-negativity inequality. To see this, note that by definition, $P(G_1), P(G_2)$ are described by non-negativity inequalities and degree equations. Thus, it can be readily checked that $P(G; C)$ is described by non-negativity inequalities, degree equations, and $x(C) = 1$, in turn implying the claim.

**4.4. Number of bricks in cut-contractions.** Suppose $x(C) \geq 1$ exposes a face of $P(G)$ of dimension $d - i$, for some integer $i \geq 0$. Then $1 + d - i = |B| = |B_1| + |B_2| - |C|$, so

$$|E| - |V| + 2 - b(G) - i = |E(G_1)| - |V(G_1)| + 2 - b(G_1) + |E(G_2)| - |V(G_2)| + 2 - b(G_2) - |C|,$$

implying in turn that $b(G_1) + b(G_2) = b(G) + i$.

**4.5. Number of bricks in cut-contractions of near-bricks.** **(1)** In general, if one of $G_1, G_2$ is bipartite, then a simple counting argument implies that $C$ is a tight cut. That is, if $C$ is not tight, then $\min\{b(G_1), b(G_2)\} \geq 1$. **(2)** The converse also holds if $G$ is a near-brick, by 4.4. **(3)** Subsequently, if $G$ is a near-brick, then $C$ is facet-defining if, and only if, both $G_1, G_2$ are near-bricks.

**4.6. Tight cuts in near-bricks and barriers.** Suppose $G$ is a near-brick, and $C$ is a tight cut of $G$. Then one of $G_1, G_2$ is a near-brick while the other one, say $G_1$, is bipartite, by 4.4. Let $B$ be the part in a bipartition of $G_1$ where $x \notin B$. Observe that $B$ is an independent set of $G$, and $G \setminus B$ has exactly $|B|$ connected components, one of which is $X$, and all others are singletons. We say that $B$ is a *barrier* in $G$, and $C$ is a *barrier cut* with barrier $B$. A barrier is *maximal* if it is not contained in another barrier.[1]

**4.7. Tight cuts in near-bricks and facet-defining cuts.** Suppose $G$ is a near-brick, $C$ is a tight cut of $G$, $G_1$ is bipartite, and $G_2$ is a non-BvN near-brick. If $D = \delta(Y)$ is a separating facet-defining cut of $G_2$ with $Y \subset X$, then $D$ is a separating facet-defining cut of $G$.

To this end, as $D$ is separating in $G_2$, and $C$ is separating in $G$, it follows that $D$ is separating in $G$. As $G_2$ is a near-brick, both $G_2/Y, G_2/\bar{Y}$ are near-bricks, by 4.5, part (3). Thus, $G/\bar{Y} = G_2/\bar{Y}$ is a near-brick. As $G_2/Y$ is a near-brick, $C = \delta(\bar{X})$ is a tight cut in $G/Y$, and $G_1$ is bipartite, it follows that $G/Y$ is a near-brick. Subsequently, both $G/Y, G/\bar{Y}$ are near-bricks, implying that $D$ is a separating facet-defining cut of $G$, by 4.5, part (3).

**4.8. Equivalent cuts in near-bricks.** Two odd cuts $C_1, C_2$ are *equivalent* if $x(C_1) = x(C_2)$ for all $x \in P(G)$. Suppose $G$ is a near-brick, and $C_1 = \delta(X_1), C_2 = \delta(X_2)$ are separating cuts that define one and the same facet, where $|X_1 \cap X_2|$ is odd. We claim that $C_1, C_2$ are equivalent; furthermore, if $X_1 \subset X_2$, then $G/X_1/\bar{X}_2$ is bipartite. To this end, note that $G/\bar{X}_1, G/\bar{X}_2$ are near-bricks.

If $X_1 = X_2$, then the claim is clear.

[1] Barriers are defined more broadly and for all matching-covered graphs, but for our purposes, we shall focus only on the (special type of) barriers defined here for near-bricks.

If $X_1, X_2$ do not cross, say $X_1 \subset X_2$, then $\delta(X_1)$ is a tight cut in the near-brick $G/\bar{X}_2$, and as $G/\bar{X}_1$ is a near-brick, it follows that $G/X_1/\bar{X}_2$ is bipartite matching-covered with $x_1, \bar{x}_2$ on opposite sides of any bipartition. A simple counting argument now implies that for every perfect matching $M$ of $G$, $|M \cap \delta(X_1)| = |M \cap \delta(X_2)|$, so $C_1, C_2$ are equivalent cuts in $G$.

Otherwise, $X_1, X_2$ cross. Let $I := \delta(X_1 \cap X_2)$ and $U := \delta(X_1 \cup X_2)$. Given that $P(G; C_1) \cap P(G; C_2) = P(G; C_1) \not\subseteq \{x : x_e = 0\}$ for any edge $e \in E$, it follows from 4.2 that $P(G; C_1) = P(G; I) \cap P(G; U)$. As $P(G; C_1)$ is a facet of $P(G)$, at least one of $I, U$ must define the same facet as $C_1$, say it is $I$. By applying the argument above to $X_1, X_1 \cap X_2$, and also to $X_2, X_1 \cap X_2$, we obtain that $C_1, I$ and $C_2, I$ are equivalent, so $C_1, C_2$ are equivalent, as claimed.

**4.9. Basis for near-bricks with a Petersen brick.** Suppose $G$ is a near-brick with a Petersen brick $H$ obtained through a tight cut decomposition. Let $Y$ be the vertex set of a 5-cycle of $H$, and let $D := \delta(Y)$. Then there exist perfect matchings $M_0, M_1, \ldots, M_d$ of $G$ whose incidence vectors form a basis for $\mathrm{lin}(P(G))$, where $|M_0 \cap D| = 5$, $|M_i \cap D| = 1$ for all $i \in [d]$, and every edge of $G$ is contained in at least one of $M_1, \ldots, M_d$.

To see this, note first that the Petersen graph $\mathbb{P}$ has exactly six perfect matchings $N_0, N_1, \ldots, N_5$, whose incidence vectors form a basis for $\mathrm{lin}(P(\mathbb{P}))$. One of the perfect matchings, say $N_0$, intersects $\delta_{\mathbb{P}}(Y)$ in five edges, while the remaining perfect matchings intersect $\delta_{\mathbb{P}}(Y)$ just once. Note further that every edge of $\mathbb{P}$ belongs to exactly two perfect matchings, so at least one of these must be among $N_1, \ldots, N_5$.

Subsequently, given that $H$ is obtained from $\mathbb{P}$ by adding some $p$ edges parallel to existing edges, we can find perfect matchings $N_6, \ldots, N_{p+5}$ of $H$ such that the incidence vectors of $N_0, N_1, \ldots, N_{p+5}$ form a basis for $\mathrm{lin}(P(H))$, $|N_0 \cap D| = 5$, and $|N_i \cap D| = 1$ for all $i \in [p]$, and every edge of $H$ belongs to at least one of $N_1, \ldots, N_{p+5}$.

The existence of $M_0, M_1, \ldots, M_d$ now follows by recursively applying 4.1, part (4).

**4.10. Cut and face triples.** A *cut triple* is a tuple $(C_1, C_2, C_3)$ of odd cuts of $G$ of the form $C_i = \delta(X_i), i \in [3]$ such that $X_1 \subset X_2 \subset X_3$. A *face triple* is a tuple $(F_1, F_2, F_3)$ of faces of $P(G)$ such that $F_2 \cap F_1 = F_2 \cap F_3$, $F_2 \cap F_3$ is not contained in $\{x : x_e = 0\}$ for any $e \in E$, and at least one of $F_2 \setminus F_1, F_2 \setminus F_3$ is nonempty. We claim that a cut triple cannot define a face triple, that is, we cannot have $P(G; C_i) = F_i, i = 1, 2, 3$.

Suppose otherwise. By symmetry between $C_1$ and $C_3$, we may assume that $F_2 \setminus F_1 \neq \emptyset$. Let $M$ be a perfect matching such that $|M \cap C_2| = 1$ and $|M \cap C_1| > 1$. Write $M \cap C_2 = \{f\}$. As $F_2 \cap F_3 \not\subseteq \{x : x_f = 0\}$, there exists a perfect matching $M'$ such that $M' \cap C_2 = \{f\}$ and $|M' \cap C_3| = 1$.

Let $M''$ be the perfect matching such that $M'' \cap C_2 = \{f\}$, and agrees with $M$ in $G[X_2]$ and with $M'$ in $G[\bar{X}_2]$. Then $|M'' \cap C_1| = |M \cap C_1| > 1$ and $|M'' \cap C_3| = |M' \cap C_3| = 1$. Subsequently, the incidence vector of $M''$ belongs to $F_2 \cap F_3$ but not $F_2 \cap F_1$, a contradiction as $F_2 \cap F_1 = F_2 \cap F_3$.

# 5 Proof of the Intersection Theorem

Let $G = (V, E)$ be a Petersen-free non-BvN near-brick. We prove by induction on $|E|$ that there exist a perfect matching $M$ and a separating facet-defining cut $C$ such that $|M \cap C| = 3$; we shall call $(M, C)$ a *desired pair*. We proceed in four stages.

**From near-bricks to bricks.** We first reduce to the case where $G$ is a brick. To this end, suppose $G$ has a tight cut, and let $G_1, G_2$ be the corresponding cut-contractions, where $G_1$ is bipartite, and $G_2$ is a Petersen-free near-brick. We know that $G_2$ is non-BvN, by 4.3. By the induction hypothesis, $G_2$ has a perfect matching $N$ and a separating facet-defining cut $C$ such that $|N \cap C| = 3$. Then $C$ is a separating facet-defining cut of $G$, by 4.7. Thus, by extending $N$ to a perfect matching $M$ of $G$, we see that $(M, C)$ is a desired pair.

**The facets.** Next we reduce to the case where no facet of $P(G)$ is exposed by a non-negativity inequality.

Suppose some facet of $P(G)$ is exposed by a non-negativity inequality. As $G$ is not BvN, there exists one such facet, say exposed by $x_e \geq 0$, that is adjacent to a facet not exposed by a non-negativity inequality.

We claim that $G \setminus e$ is a non-BvN near-brick. To this end, let $F$ be the union of all perfect matchings of $G \setminus e$. Then $G|F$ is a matching-covered graph such that $P(G|F)$ is obtained from the facet $P_e(G)$ after removing the coordinates that are equal to 0 throughout the facet. Our choice of the facet $P_e(G)$ implies that $P(G|F)$ has a facet not exposed by a non-negativity inequality, so $G|F$ is not BvN, implying in turn that $G|F$ is not bipartite, so $b(G|F) \geq 1$. On the other hand,

$$d - 1 = \dim(P(G|F)) = |F| - |V| + 1 - b(G|F) = d + 1 - |E \setminus F| - b(G|F),$$

implying in turn that $|E \setminus F| + b(G|F) = 2$, so $|E \setminus F| = b(G|F) = 1$. Thus, $G|F = G \setminus e$ is a non-BvN near-brick.

Suppose $G \setminus e$ has a Petersen brick. Let $H$ be the Petersen brick of $G \setminus e$, obtained through a tight cut decomposition of $G \setminus e$. We have the following three cases.

Case 1. $H = G \setminus e$. In this case, as $G$ is a non-Petersen brick, $e$ must connect a pair of vertices of $H$ at distance 2.

Otherwise, $H \neq G \setminus e$, so $G \setminus e$ has a tight cut. Every tight cut of $G \setminus e$ is a barrier cut of the form $\delta_{G \setminus e}(X)$ with barrier $B \subset \bar{X}$, by 4.6. Observe that $P_e(G) \subseteq P(G; \delta_G(X))$, and since $P(G; \delta_G(X)) \neq P(G)$ as $G$ is a brick, we must have $P_e(G) = P(G; \delta_G(X))$. A simple counting argument implies that $e$ joins either two singleton components of $G \setminus e \setminus B$, or it joins a singleton component to $X$. This implies that there exist either one or two maximal barriers only, summarized as follows.

Case 2. $H = G \setminus e / \bar{X}$, where $\delta_{G \setminus e}(X)$ is a barrier cut with barrier $B \subset \bar{X}$, and $B$ is the unique maximal barrier of $G \setminus e$.

Case 3. $H = G \setminus e / \bar{X}_1 / \bar{X}_2$, where $\delta_{G \setminus e}(X_i)$ is a barrier cut with barrier $B_i \subset \bar{X}_i$ for $i \in [2]$, $\bar{X}_1 \cap \bar{X}_2 = \emptyset$, and $e$ joins a singleton component of $G \setminus e \setminus B_1$ to a singleton component of $G \setminus e \setminus B_2$. Furthermore, $B_1, B_2$ are the only maximal barriers of $G \setminus e$.

It can be shown through an elementary though terse argument that in each of Cases 1-3, there exist a 5-cycle of $H$ with vertex set $Y$, and perfect matchings $M, M'$ of $G$ containing $e$, such that $Y$ does not use contraction vertices of $H$, $Y$ is not incident to either ends of $e$, $|M \cap \delta_G(Y)| = 3$, and $|M' \cap \delta_G(Y)| = 1$.[2] Moving on, we claim that $\delta_G(Y)$ is a separating facet-defining cut of $G$. Clearly,

[2] For a detailed analysis, we refer the reader to the proof of ([dCLM02a], Theorem 5.4), or ([LM24], Theorem 13.6).

$\delta_{G\setminus e}(Y)$ is separating in $G \setminus e$ as it is so in $H$. Thus, as $e \in M'$ and $|M' \cap \delta_G(Y)| = 1$, it follows that $\delta_G(Y)$ is separating in $G$. To see that $\delta_G(Y)$ is facet-defining in $G$, pick perfect matchings $M_1, \dots, M_d$ whose incidence vectors lie inside the facet $P_e(G)$ and are linearly independent; we may pick the first $d-1$ perfect matchings to have intersection 1 with $\delta_G(Y)$, and the last one with intersection 5, by 4.9. We then swap $M_d$ with the perfect matching $M' \ni e$ which satisfies $|M' \cap \delta(Y)| = 1$, in order to obtain $d$ linearly independent vectors $M_1, \dots, M_{d-1}, M'$ in the face $P(G; \delta_G(Y))$, implying in turn that $\delta_G(Y)$ defines a facet of $P(G)$. Consequently, $(M, \delta_G(Y))$ is a desired pair.

We may therefore assume that $G \setminus e$ is a Petersen-free non-BvN near-brick. By the induction hypothesis, there exist a perfect matching $M$ and a separating facet-defining cut $\delta_{G\setminus e}(X)$ of $G \setminus e$ such that $|M \cap \delta_{G\setminus e}(X)| = 3$. Let $C := \delta_G(X)$. Observe that $P(G; C)$ is either facet of $P(G)$, or a $(d-2)$-face of $P(G)$ contained in $P_e(G)$. Observe further that $P(G; C)$ is not contained in $\{x : x_f = 0\}$ for any $f \in E \setminus e$.

Suppose first that $P(G; C)$ is a facet of $P(G)$. Given that $e \notin M$ and $|M \cap C| \neq 1$, it follows that $P(G; C) \neq P_e(G)$. Subsequently, $P(G; C)$ is not contained in $\{x : x_f = 0\}$ for any edge $f \in E$, so $C$ is a separating facet-defining cut of $G$. Subsequently, $(M, C)$ is a desired pair.

In the remaining case, $P(G; C)$ is a $(d-2)$-face of $P(G)$, contained in the facet $P_e(G)$. Consider now the other facet of $P(G)$ containing $P(G; C)$. It cannot be exposed by a non-negativity inequality, given that $P(G; C)$ is not contained in $\{x : x_f = 0\}$ for any $f \in E \setminus e$. In particular, the other facet is exposed by a separating facet-defining cut of $G$, say $D$. In particular, $D \setminus e$ defines the same facet as $C \setminus e$, and so $D \setminus e, C \setminus e$ must be equivalent cuts as $G \setminus e$ is a near-brick, by 4.8. Thus, $|M \cap D| = |M \cap C| = 3$, implying in turn that $(M, D)$ is a desired pair.

**The $(d-2)$-faces.** Thus every non-negativity inequality defines a face of $P(G)$ of dimension at most $d-2$. In this stage, we reduce to the case where every $(d-2)$-face of $P(G)$ is exposed by a non-negativity inequality.

Suppose some $(d-2)$-face of $P(G)$ is not exposed by a non-negativity inequality. Take a facet containing this face, which is exposed by a separating facet-defining cut $C = \delta(X)$. Then both $C$-contractions of $G$ are near-bricks, by 4.5, part (3). Given that the polytope $P(G; C)$ has a facet not exposed by a non-negativity inequality, one of the $C$-contractions, say $G/\bar{X}$, must be non-BvN, by 4.3.

We may assume that $G/\bar{X}$ is a non-BvN brick. To argue this, note first that the unique brick of $G/\bar{X}$ is non-BvN, by 4.3. If $G/\bar{X}$ is not this brick, then choose a minimal vertex subset $Y \subset X$ such that $\delta(Y)$ is a tight cut of $G/\bar{X}$. In particular, $P(G; C) \subseteq P(G; \delta(Y))$. As $G$ is a brick, $\delta(Y)$ is not tight in $G$, so $P(G; C) = P(G; \delta(Y))$, therefore $C, \delta(Y)$ are equivalent cuts and $G/Y/\bar{X}$ is bipartite, by 4.8. This, together with our minimal choice of $Y$, implies that $G/\bar{X}/\bar{Y} = G/\bar{Y}$ is a brick. Subsequently, by changing $C = \delta(X)$ and without changing the corresponding facet, if necessary, we may assume $G/\bar{X}$ is a non-BvN brick.

If $G/\bar{X}$ is a Petersen brick, then any perfect matching $M$ intersecting $C$ more than once must intersect $C$ exactly three times, so $(M, C)$ is a desired pair.

Therefore, we may assume that $G/\bar{X}$ is a non-Petersen non-BvN brick. By the induction hypothesis, $G/\bar{X}$ has a separating facet-defining cut $R$ and a perfect matching $N$ such that $|R \cap N| = 3$. Extend $N$ to a perfect matching $M$ of $G$. Then $|R \cap M| = 3$ and $|C \cap M| = 1$. Clearly, $R$ is a separating cut of $G$. Observe that $P(G; R)$ is either a facet, or a $(d-2)$-face of $P(G)$ contained in $P(G; C)$.

If $P(G;R)$ is a facet of $P(G)$, then $(M,R)$ is a desired pair.

Otherwise, $P(G;R)$ is a $(d-2)$-face of $P(G)$ contained in $P(G;C)$. Let $P(G;D)$ be the other facet of $P(G)$ containing $P(G;R)$. Observe that $x(R) \geq 1$ and $x(D) \geq 1$ define the same facet of the polytope $P(G;C)$. Observe further that $(P(G;R), P(G;C), P(G;R))$, as well as any permutation of $(P(G;R), P(G;C), P(G;D))$ without $P(G;R)$ in the middle, is a face triple.

We shall prove that $(M,D)$ is a desired pair. To this end, write $D = \delta(Z)$ where $Z \subset V$ and $|X \cap Z|$ is odd. There are three cases:

Case 1. Suppose $Z \subset X$. Then $D$ is an odd cut of $G/\bar{X}$, so $R,D$ define the same facet of $P(G/\bar{X})$, so $R,D$ are equivalent cuts of $G/\bar{X}$ as $G/\bar{X}$ is a brick, by 4.8. Thus, $|D \cap N| = |R \cap N| = 3$, so $|D \cap M| = |D \cap N| = 3$, therefore $(M,D)$ is a desired pair.

Case 2. Suppose $X \subset Z$. Then $(R,C,D)$ would be a cut triple defining a face triple, thus contradicting 4.10.

Case 3. In the remaining case, $X$ and $Z$ cross. Let $I := \delta(X \cap Z)$ and $U := \delta(X \cup Z)$. Given that $P(G;C) \cap P(G;D) = P(G;R) \not\subseteq \{x : x_e = 0\}$ for any $e \in E$, it follows from 4.2 that $x(C) + x(D) = x(I) + x(U)$ and $P(G;R) = P(G;C) \cap P(G;D) = P(G;I) \cap P(G;U)$.

If $\{P(G;I), P(G;U)\} = \{P(G;R), P(G;C)\}$ or $\{P(G;R), P(G;D)\}$, then $(I,D,U)$ or $(I,C,U)$ would be a cut triple defining a face triple, respectively, thus contradicting 4.10.

If $\{P(G;I), P(G;U)\} = \{P(G;R), P(G)\}$, then we have $P(G;I) = P(G;R)$ and $P(G;U) = P(G)$. This holds since otherwise $(R,C,U)$ would be a cut triple defining a face triple, thus contradicting 4.10. The equality $P(G;U) = P(G)$ implies that $U$ is a trivial cut as $G$ is a brick, while $P(G;I) = P(G;R)$ implies that $R,I$ are equivalent cuts in the brick $G/\bar{X}$, so $x(I) = x(R)$ for all $x \in P(G)$. Consequently,

$$x(D) = x(I) + x(U) - x(C) = x(R) + 1 - x(C),$$

so whenever $x(C) = 1$, then $x(R) = x(D)$, so $|D \cap M| = |R \cap M| = 3$, so $(M,D)$ is a desired pair.

Otherwise, $\{P(G;I), P(G;U)\} = \{P(G;C), P(G;D)\}$, then we have $P(G;I) = P(G;D)$ and $P(G;U) = P(G;C)$. This holds since otherwise $(R,C,U)$ would be cut triple defining a face triple, thus contradicting 4.10. Now by replacing $D$ with $I$ we fall back to Case 1.

**The finale.** We have reduced to the case where $G$ is a non-BvN non-Petersen brick where every $(d-2)$-face of $P(G)$ is exposed by a non-negativity inequality. The existence of a desired pair now follows from the Petersen Graph Lemma, thus finishing the proof. □

## 6 Proof of the Main Theorem

In this final section, we prove the Integral Basis Theorem, and then obtain the Main Theorem as a corollary. For a polyhedron $P \subseteq \mathbb{R}^n$ and $k \geq 0$, define $kP$ as the set of all points of the form $\sum_{p \in P} \lambda_p p$ where $\lambda \in \mathbb{R}^P_{\geq 0}$ and $\mathbf{1}^\top \lambda = k$. Here, $\mathbf{1}$ denotes the all-ones vector of appropriate dimension. $P$ has the *integer decomposition property* if for every integer $k \geq 1$, every integral point in $kP$ can be written as the sum of $k$ integral points in $P$. We need the following preliminary.

**Theorem 6.1** ([ACLS25]). *Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron with the integer decomposition property, whose affine hull is of the form $\{x : Ax = b\}$ for $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ such that $m \geq 1$, $b \neq \mathbf{0}$, and $\gcd\{b_i : i \in [m]\} = 1$. Then $P \cap \mathbb{Z}^n$ contains an integral basis for $\operatorname{lin}(P)$.*

We are now ready to prove the Integral Basis Theorem.

*Proof of Theorem 1.2.* We prove this by induction on $|V|$. Let $P := P(G)$.

**Base case.** Suppose $G$ is BvN. We claim that $P$ has the integer decomposition property. To this end, let $x \in kP \cap \mathbb{Z}^E$ for some integer $k \geq 1$, that is, $x \in \mathbb{Z}^E_{\geq 0}$ and $x(\delta(v)) = k$, $\forall v \in V$. If $k = 1$, then $x$ is the incidence vector of a perfect matching. Otherwise, as $\frac{1}{k}x \in P$ and $P$ is an integral polytope, we can express $\frac{1}{k}x$ as a convex combination of the vertices of $P$, each of which is an incidence vector of a perfect matching of $G$. In particular, there is a perfect matching $M$ such that $\mathbf{1}_M \leq x$, where $\mathbf{1}_M \in \{0,1\}^E$ is the incidence vector of $M \subseteq E$. Let $x' := x - \mathbf{1}_M$, which satisfies $x' \in \mathbb{Z}^E_{\geq 0}$ and $x'(\delta(v)) = k - 1$, $\forall v \in V$. By repeating this argument, we obtain a description of $x$ as the sum of $k$ vectors, each of which is an incidence vector of a perfect matching of $G$. Subsequently, $P$ has the integer decomposition property. As $G$ is matching-covered, the affine hull of $P$ is $\{x : x(\delta(v)) = 1, \forall v \in V\}$. Given that the GCD of the right-hand side values is 1, it follows from Theorem 6.1 that $P \cap \mathbb{Z}^E$ contains an integral basis for $\operatorname{lin}(P)$, as required.

**Tight cuts.** Suppose there is a tight cut $C$. Let $G_1, G_2$ be the $C$-contractions of $G$, both of which are Petersen-free matching-covered graphs. By the induction hypothesis, for each $i \in [2]$, $\operatorname{lin}(P(G_i))$ has an integral basis $B^i$ that consists solely of the incidence vectors of some perfect matchings of $G_i$. Let $B := B^1 \odot B^2$ which is a subset of $\mathbb{Z}^E$ consisting of incidence vectors of some perfect matchings of $G$ that intersect $C$ just once, and is an integral basis for $\operatorname{lin}(P(G; C))$, by 4.1, part (3). Given that $C$ is a tight cut, it follows that $P(G; C) = P$, thus completing the induction step.

**Non-BvN brick.** Otherwise, $G$ is a non-BvN non-Petersen brick. By the Intersection Theorem, there exist a separating facet-defining cut $C = \delta(X)$ and a perfect matching $M$ such that $|C \cap M| = 3$. Let $G_1 := G/X$ and $G_2 := G/\bar{X}$, both of which are near-bricks, by 4.5, part (3).

Suppose in the first case that both $G_1, G_2$ are Petersen-free. Then by the induction hypothesis, for each $i \in [2]$, $\operatorname{lin}(P(G_i))$ has an integral basis $B^i$ that consists solely of the incidence vectors of some perfect matchings of $G_i$. Let $B' := B^1 \odot B^2$ which is a subset of $\mathbb{Z}^E$ consisting of incidence vectors of some perfect matchings of $G$ that intersect $C$ just once, and is an integral basis for $\operatorname{lin}(P(G; C))$, by 4.1, part (3). We claim that $B := B' \cup \{\mathbf{1}_M\}$ is an integral basis for $\operatorname{lin}(P)$. Linear independence is clear as $|M \cap C| > 1$ while $b(C) = 1$ for all $b \in B'$. Given that $\dim(P) = \dim(P(G; C)) + 1$, it therefore follows that $B$ is a basis for $\operatorname{lin}(P)$. It remains to prove that $B$ is an *integral* basis. To this end, let $z \in \operatorname{lin}(P) \cap \mathbb{Z}^E$, and express this vector as a unique linear combination of $B$: $z = \sum_{b\in B} \alpha_b b$. We need to show that $\alpha_b \in \mathbb{Z}$ for all $b \in B$.

First we show that $\alpha_M := \alpha_{\mathbf{1}_M}$ is an integer. As $z \in \operatorname{lin}(P)$, it follows that $z(\delta(u)) = z(\delta(v))$ for all $u, v \in V$. Let $c := z(\delta(v)) \in \mathbb{Z}$. Recall that $C = \delta(X)$ for an odd-sized $X \subset V$. We have

$$|X| \cdot c = \sum_{v\in X} z(\delta(v)) = z(C) + 2z(E[X]),$$

so $c \equiv z(C) \pmod 2$. Given $v \in V$, we have $c = z(\delta(v)) = \sum_{b \in B} \alpha_b b(\delta(v)) = \sum_{b \in B} \alpha_b$, as each $b \in B$ is the incidence vector of a perfect matching. Thus, as $b(C) = 1$ for all $b \in B'$ and $|M \cap C| = 3$,

$$z(C) = \sum_{b \in B} \alpha_b b(C) = 2\alpha_M + \sum_{b \in B} \alpha_b = 2\alpha_M + c.$$

Hence, $2\alpha_M = z(C) - c$, which is an even integer. Subsequently, $\alpha_M \in \mathbb{Z}$.

Thus, $z - \alpha_M \mathbf{1}_M$ is an integral vector in $\mathrm{lin}(B')$, so given that $B'$ is an integral basis for its linear hull, it follows that $\alpha_b \in \mathbb{Z}$ for all $b \in B'$, as desired.

Suppose in the remaining case that (at least) one of $G_1, G_2$, say $G_2 = G/\bar{X}$, has a Petersen brick. We will adjust our choice of $C$ so that we fall in the previous case.

We may assume that the near-brick $G_2$ is a Petersen brick. If not, then choose a minimal subset $Z \subset X$ such that $\delta(Z)$ is a tight cut of $G/\bar{X}$. In particular, $P(G; C) \subseteq P(G; \delta(Z))$. As $G$ is a brick, $\delta(Z)$ is not tight in $G$, so $\delta(Z)$ and $C$ define the same facet of $P$, so they are equivalent cuts and $G/Z/\bar{X}$ is bipartite, by 4.8. This, together with our minimal choice of $Z$, implies that $G/\bar{Z}$ is a brick. Subsequently, by changing $C = \delta(X)$ to an equivalent cut if necessary, we may assume $G_2$ is a Petersen brick.

Let $Y \subset X$ be the vertex set of a 5-cycle of $G_2$. We claim that $\delta(Y)$ is a separating facet-defining cut of $G$ whose cut-contractions are Petersen-free.

Given that $\delta(Y)$ is separating in $G_2$ and $\delta(X)$ is separating in $G$, it follows that $\delta(Y)$ is separating in $G$.

To argue that $\delta(Y)$ is facet-defining for $G$, note first that the perfect matching $M$ can be redefined inside $G[X]$, if necessary, such that $|M \cap \delta(Y)| = 1$. Secondly, by 4.1, part (4) and 4.9, there exist perfect matchings $M_1, \ldots, M_d$ whose incidence vectors belong to $P(G; C)$ and form a basis for $\mathrm{lin}(P(G; C))$, where $|M_1 \cap \delta(Y)| = 5$, $|M_i \cap \delta(Y)| = 1$ for $i \in \{2, \ldots, d\}$, and every edge of $G$ belongs to one of $M_2, \ldots, M_d$. By swapping $M_1$ with $M$, we obtain $d$ perfect matchings $M, M_2, \ldots, M_d$ whose incidence vectors belong to $P(G; \delta(Y))$ and are linearly independent, because $|M \cap C| = 3$ while $|M_i \cap C| = 1$ for $i \in \{2, \ldots, d\}$. Thus, $P(G; \delta(Y))$ is a facet of $P$.

Finally, for each $i \in [2]$, the near-brick $G_i$ must be Petersen-free because it contains a triangle, and such a triangle also belongs to the unique brick of $G_i$. □

We are now ready to prove the Main Theorem.

*Proof of Theorem 1.1.* We proceed by induction on $|V|$. For the base case, suppose $G$ is a brick. If $G$ is a non-Petersen brick, then the result follows from Theorem 1.2. Otherwise, $G$ is a Petersen brick, and the result can be readily checked.

For the induction step, suppose $C$ is a tight cut of $G$, and let $G_1, G_2$ be the $C$-contractions, where $G_1$ has Petersen bricks $H_1, \ldots, H_q$, and $G_2$ has Petersen bricks $H_{q+1}, \ldots, H_p$, obtained through tight cut decompositions of $G_1, G_2$, respectively. Let $L_i := L(G_i)$ and $\bar{L}_i := \mathrm{lin}(P(G_i)) \cap \mathbb{Z}^{E(G_i)}$, for $i \in [2]$. By the induction hypothesis, $L_i$ has a lattice basis $B_i$ consisting solely of some perfect matchings of $G_i$, for $i = 1, 2$. Furthermore,

$$\begin{aligned} L_1 &= \bar{L}_1 \cap \{x : x(A_i) \equiv 0 \pmod 2,\ i = 1, \ldots, q\}, \\ L_2 &= \bar{L}_2 \cap \{y : y(A_i) \equiv 0 \pmod 2,\ i = q+1, \ldots, p\}, \end{aligned}$$

where each $A_i$ is the edge set of some 5-cycle of $H_i$. Let $B := B_1 \odot B_2$, which clearly consists of some perfect matchings of $G$. By 4.1, part (2), $B$ is a lattice basis for $L$. Furthermore, we claim that the above set equalities imply

$$L = \bar{L} \cap \{z : z(A_i) \equiv 0 \pmod 2, \ \forall i \in [p]\}.$$

($\subseteq$) is clear. ($\supseteq$): Pick $f \in \bar{L}$ such that $f(A_i) \equiv 0 \pmod 2, \ \forall i \in [p]$. Then $f = x \odot y$ for $x \in \bar{L}_1$ and $y \in \bar{L}_2$, where clearly $x(A_i) \equiv 0 \pmod 2, \ i = 1, \ldots, q$ and $y(A_i) \equiv 0 \pmod 2, \ i = q+1, \ldots, p$. Thus, $x \in L_1, y \in L_2$, so $f$ belongs to $L$, by 4.1, part (2). This completes the induction step. $\square$

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## A Details about the composition procedure

Let us provide a proof of 4.1, parts (1)-(3). Our proof is very similar, and at times identical, to that of ([ACLS25], Lemma 5.1). First, it is clear from the construction that $B \subseteq P(G;C) \cap \{0,1\}^E$, and $|B| = |B_1| + |B_2| - |C|$. We need the following two claims.

**A.1.** $B$ is linearly independent.

*Proof of A.1.* To prove the linear independence of $B$, suppose $\sum_{e,i} \lambda_i^e z_i^e = 0$ for some $\lambda_i^e \in \mathbb{R}$ for all $e \in C, 1 \leq i \leq |I_e| + |J_e| - 1$. Fix $e \in C$ with $I_e = \{i_1, \ldots, i_k\}$ and $J_e = \{j_1, \ldots, j_\ell\}$. Given that $B_1$ is linearly independent, then for each $x^{i_t}$, the sum of the coefficients of vectors in $B$ of the form $x^{i_t} \odot y$ for some $y$, must be $0$. Subsequently, we have

$$\sum_{i=1}^{\ell} \lambda_i^e = 0 \tag{1}$$

$$\lambda_{\ell+1}^e = \cdots = \lambda_{\ell+k-1}^e = 0 \tag{2}$$

where (1) computes the coefficient for $x^{i_1} \odot y$, while (2) computes the coefficients for $x^{i_t} \odot y, t = 2, \ldots, k$. Similarly, given that $B_2$ is linearly independent, for each $y^{j_t}$, the sum of the coefficients of vectors in $B$ of the form $x \odot y^{j_t}$ for some $x$, must be $0$. Subsequently, we obtain that

$$\sum_{i=\ell}^{\ell+k-1} \lambda_i^e = 0 \tag{3}$$

$$\lambda_2^e = \cdots = \lambda_\ell^e = 0 \tag{4}$$

where (3) computes the coefficient for $x \odot y^{j_1}$, while (4) computes the coefficients for $x \odot y^{j_t}, t = 2, \ldots, \ell$. Observe that (1) and (4) imply that $\lambda_1^e = 0$, so together with (2), we obtain that $\lambda_i^e = 0$ for all $1 \leq i \leq k + \ell - 1$. As this holds for all $e \in C$, $\lambda_i^e = 0$ for all $e \in C, 1 \leq i \leq |I_e| + |J_e| - 1$. $\Diamond$

**A.2.** If $x$ is an (integer) linear combination of the vectors in $B_1$ and $y$ of $B_2$, where $x_e = y_e$, $\forall e \in C$, then $x \odot y$ is an (integer) linear combination of the vectors in $B$.

*Proof of A.2.* Suppose $x = \sum_i \alpha(x^i)x^i$ and $y = \sum_j \beta(y^j)y^j$ for real numbers $\alpha(x^i)$ and $\beta(y^j)$. Fix $e \in C$ with $I_e = \{i_1, \ldots, i_k\}$ and $J_e = \{j_1, \ldots, j_\ell\}$. Now choose $\lambda_i^e$ for all $1 \leq i \leq k + \ell - 1$ such that

$$\sum_{i=1}^{\ell} \lambda_i^e = \alpha(x^{i_1}) \tag{5}$$

$$\lambda_{\ell+t-1}^e = \alpha(x^{i_t}) \quad t = 2, \ldots, k \tag{6}$$

$$\lambda_t^e = \beta(y^{j_t}) \quad t = 2, \ldots, \ell. \tag{7}$$

(6) and (7) give us the values for $\lambda_t^e, t = 2, \ldots, \ell + k - 1$. Furthermore, (5) and (7) give us

$$\lambda_1^e = \alpha(x^{i_1}) - \sum_{t=2}^{\ell} \beta(y^{j_t}).$$

Since $x_e = y_e$, it can be readily checked that $\alpha(x^{i_1}) - \sum_{t=2}^{\ell} \beta(y^{j_t}) = \beta(y^{j_1}) - \sum_{t=2}^{k} \alpha(x^{i_t})$, so

$$\sum_{i=\ell}^{\ell+k-1} \lambda_i^e = \beta(y^{j_1}). \tag{8}$$

It follows from (5)-(8) that $x \odot y = \sum_{e,i} \lambda_i^e z_i^e$. Observe that if the $\alpha(x^i)$ and $\beta(y^j)$ are integral, then so are $\lambda_t^e, t = 1, 2, \ldots, \ell + k - 1$. $\Diamond$

Let $f \in P(G; C) \cap \{0,1\}^E$. Then $f = x \odot y$ where $x \in P(G_1) \cap \{0,1\}^{E(G_1)}, y \in P(G_2) \cap \{0,1\}^{E(G_2)}$. As $B_i$ is a basis for $\operatorname{lin}(P(G_i))$ for $i \in [2]$, it follows that $x$ ($y$) is a linear combination of the vectors in $B_1$ ($B_2$), so by A.2, $f = x \odot y$ is a linear combination of the vectors in $B$. This, together with A.1, implies that $B$ is a basis for $\operatorname{lin}(P(G; C))$, thus finishing the proof for 4.1, part (1). Furthermore, if $B_i$ is a lattice basis for $L(G_i)$ for $i \in [2]$, then $x$ ($y$) is an integer linear combination of the vectors in $B_1$ ($B_2$), so by A.2, $f = x \odot y$ is an integer linear combination of the vectors in $B$. In this case, $B$ is a lattice basis for the lattice generated by $P(G; C) \cap \{0,1\}^E$, in turn proving 4.1, part (2).

Finally, assume that $B_i$ is an integral basis for $\operatorname{lin}(P(G_i))$ for $i \in [2]$. Take $f \in \operatorname{lin}(B) \cap \mathbb{Z}^E$. Observe that $f = x \odot y$, where $x \in \operatorname{lin}(B_1) \cap \mathbb{Z}^{E(G_1)}$ and $y \in \operatorname{lin}(B_2) \cap \mathbb{Z}^{E(G_2)}$. As $B_i$ is an integral basis, $x$ ($y$) must be an integer linear combination of the vectors in $B_1$ ($B_2$), so by A.2, $f = x \odot y$ is an integer linear combination of the vectors in $B$. Thus, $B$ is an integral basis for its linear hull, in turn proving 4.1, part (3). $\square$