MA431 Lecture 7

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Outline

1. The Transfer-Current Theorem
2. Weights as conductances, deletion, and contraction
3. Rayleigh monotonicity principle (weighted extension)

Electrical flow

Let $b \in \mathbb{R}^V$ be a demand vector such that $b = \nabla f$ for some flow $f \in \mathbb{R}^{\vec{E}}$.

Theorem

Let $\iota := P_\star f$. Then ι is the unique optimal solution to

$$\min \left\{ \|f'\|^2 : \nabla f' = b, f' \in \mathbb{R}^{\vec{E}} \right\}.$$

$$= \sum f_e'^2$$

Recall

ι is called the electrical flow satisfying demands b.

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Recall

ι is called the electrical flow satisfying demands b.

Remark

ι is the unique flow in $W^\star$ satisfying the demands b.

Matrix of projection onto $W^\star$

Recall

$P_\star : \mathbb{R}^{\vec{E}} \to \mathbb{R}^{\vec{E}}$ is the linear operator that projects orthogonally onto $W^\star$.

Let Π be the $\vec{E} \times \vec{E}$ matrix where column e is precisely $P_\star \chi^e$.

Remark

Π represents the linear operator $P_\star$ in the basis $\{\chi^e : e \in \vec{E}\}$.

Remark

Π is an orthogonal projection matrix, so $\Pi^2 = \Pi$ and $\Pi^\top = \Pi$.

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Kirchhoff's effective resistance theorem (rephrased)

If T is a uniformly random spanning tree, then $\Pr[e \in T] = \Pi_{e,e}$.

The Transfer-Current Theorem

The Transfer-Current Theorem

If T is a uniformly random spanning tree, then for any $F \subseteq \vec{E}$,

$$\Pr[F \subseteq T] = \det(\Pi_F)$$

where Π_F denotes the principal submatrix of Π indexed by F.

Proof by induction on $|F|$.

- Base case: $|F| = 1$.

This holds by Kirchhoff's effective resistance thm.

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Induction step: $|F| \geq 2$.

- Case 1: F contains a cycle, C. Then $\Pr[F \subseteq T] = 0$ clearly.

Then $\chi(C)$ is a circulation of $\vec{G}$ whose support is in F.

We have $\Pi\,\chi(C) = \mathbb{0}$. Thus, the columns of Π corresponding to F are lin. dep. Thus, $\det(\Pi_F) = 0$.

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Induction step: $|F| \geq 2$.

- Case 2: F is a forest. Let $e = (s, t) \in F$ and $\hat{F} = F \setminus e$. Then

$$\Pr[F \subseteq T] = \frac{T(G/F)}{T(G)} = \underbrace{\frac{T(G/\hat{F}/e)}{T(G/\hat{F})}}_{\text{the eff. res. of } e \text{ in } G/\hat{F}} \cdot \underbrace{\frac{T(G/\hat{F})}{T(G)}}_{\det(\Pi_{\hat{F}}) \text{ by IH}} = \hat{i}^e_e \cdot \det(\Pi_{\hat{F}})$$

where $\hat{i}^e$ is the elec. unit flow from s to t in $G/\hat{F}$.

The Transfer-Current Theorem

For each $a = (u, v) \in \vec{E}$, denote by i^a the unit electrical flow in G from u to v.

Lemma

$\hat{i}^e = i^e - \sum_{a \in \hat{F}} \alpha_a i^a$ for some multipliers $\alpha_a \in \mathbb{R}$.

Note. $\hat{i}^e \in \mathbb{R}^{\vec{E} \backslash \hat{F}}$. For the above equality to make sense, we extend $\hat{i}^e$ to a vector $\hat{i}^e \in \mathbb{R}^{\vec{E}}$ by appending 0s to it.

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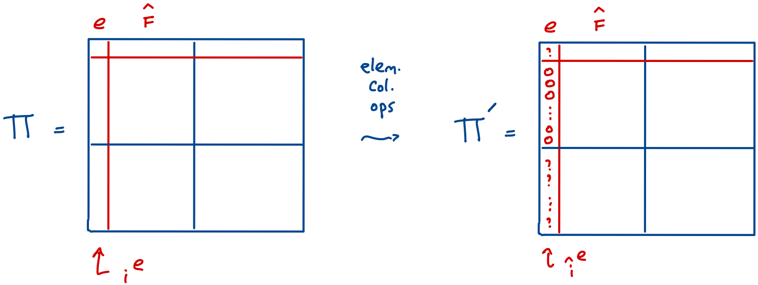

The Transfer-Current Theorem

Laplace expansion along first column

$$\det(\Pi_F) \boxed{=} \det(\Pi'_F) \boxed{=} \hat{i}_e^e \det(\Pi'_{\hat{F}}) \boxed{=} \hat{i}_e^e \det(\Pi_{\hat{F}})$$

elem.
col. ops.
don't change det.

!

b/c $\Pi'_{\hat{F}} = \Pi_{\hat{F}}$

$$= \Pr[F \subseteq T]$$

This completes the IS.

Proof of Lemma

For each $a = (u, v) \in \vec{E}$, denote by i^a the unit electrical flow in G from u to v.

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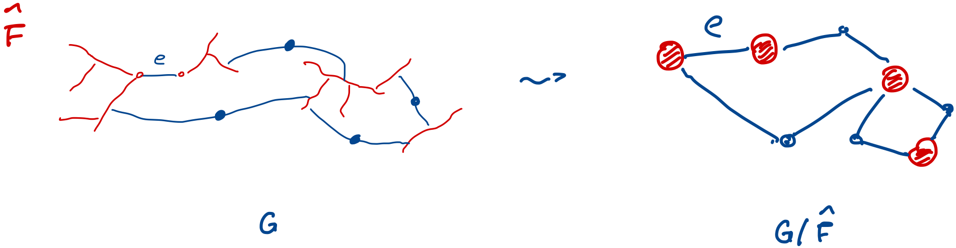
$\hat{i}^e = i^e - \sum_{a \in \hat{F}} \alpha_a i^a$ for some multipliers $\alpha_a \in \mathbb{R}$.

Proof of Lemma.

- The cuts in $G/\hat{F}$ are precisely the cuts in G that do not contain any edge in $\hat{F}$.



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Proof of Lemma.

- The cuts in $G/\hat{F}$ are precisely the cuts in G that do not contain any edge in $\hat{F}$.
- Thus the cut space $\hat{W}^\star$ of $G/\hat{F}$ is $W^\star \cap \{x : x_a = 0 \ \forall a \in \hat{F}\}$ after dropping $x_a, a \in \hat{F}$.

Proof of Lemma

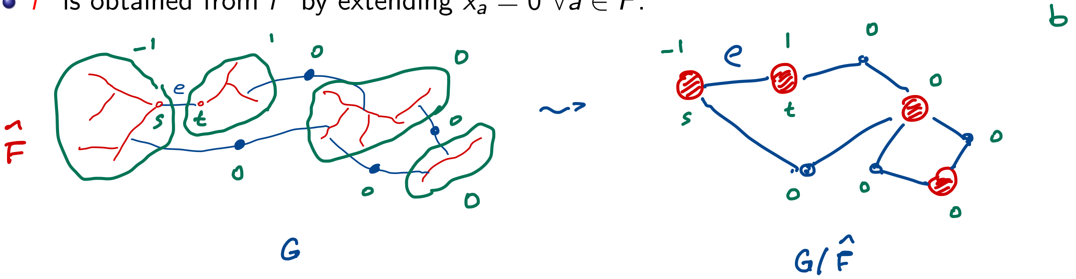
For each $a = (u, v) \in \vec{E}$, denote by i^a the unit electrical flow in G from u to v.

Lemma

$\hat{i}^e = i^e - \sum_{a \in \hat{F}} \alpha_a i^a$ for some multipliers $\alpha_a \in \mathbb{R}$.

Proof of Lemma.

- $\hat{i}^e$ is the unit flow from s to t in $G/\hat{F}$ that belongs to $\hat{W}^\star$.
- $\hat{i}^e$ is obtained from $\hat{i}^e$ by extending $x_a = 0 \ \forall a \in \hat{F}$.



Proof of Lemma

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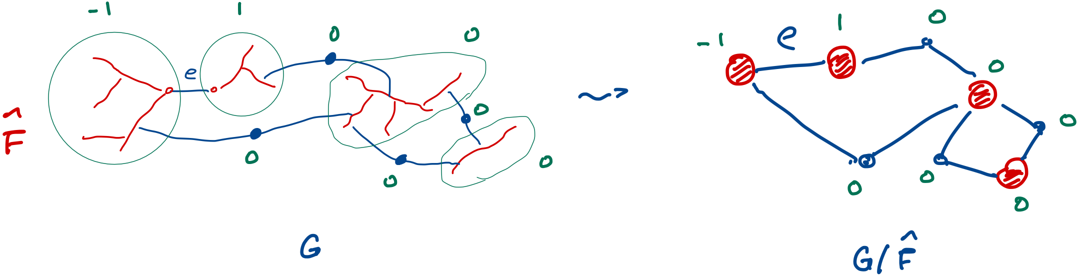
Proof of Lemma.

- In summary, $\hat{i}^e$ is *the* flow $f \in W^\star \cap \{x : x_a = 0 \; \forall a \in \hat{F}\}$ in G such that

  (P) $\sum_{v \in K} \nabla f_v = 0$ *for each connected component* K *of* $(V, \hat{F})$ *not containing* s *or* t, $\sum_{v \in K} \nabla f_v = -1$ *for the connected component* K *containing* s, *and* $\sum_{v \in K} \nabla f_v = 1$ *for the connected component* K *containing* t.

Proof of Lemma



Proof of Lemma

- We now exhibit a way to get from i^e and $i^a, a \in \hat{F}$ to $\hat{i}^e$.
- i^e is the unit flow from s to t in G that belongs to $W^\star$.
- If $i_a^e = 0\ \forall a \in \hat{F}$, then we are done.

Proof of Lemma

- We now exhibit a way to get from i^e and $i^a, a \in \hat{F}$ to $\hat{i}^e$.
- i^e is the unit flow from s to t in G that belongs to $W^\star$.
- If $i^e_a = 0 \ \forall a \in \hat{F}$, then we are done.
- Otherwise, we cancel out those flow values by considering

$$f := i^e - \sum_{a \in \hat{F}} \alpha_a i^a \in W^\star \cap \{x : x_a = 0 \ \forall a \in \hat{F}\}$$

for $\alpha_a \in \mathbb{R}_a$, $a \in \hat{F}$. (Why can this be done? Exercise.)

- Then

$$\nabla f = \nabla i^e - \sum \alpha_a \nabla i^a$$

$$= (e_t - e_s) - \sum_{a \in \hat{F}} \alpha_a (e_{h(a)} - e_{t(a)})$$

- This satisfies (P).

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Weighted extension

- Let $\vec{G} = (V, \vec{E})$ be an electrical network.
- Suppose each arc $a \in \vec{E}$ has conductance $w_a \geq 0$ and so resistance $\frac{1}{w_a} \geq 0$ ($\frac{1}{0} := \infty$).

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Intuition

By increasing the conductance of an arc, and thus decreasing its resistance, a flow now requires less energy to traverse through the arc.

In particular,

- Deleting e "corresponds to" setting $w_a = 0$
- Contracting e "corresponds to" setting $w_a = \infty$

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Energy

Given a flow $f \in \mathbb{R}^{\vec{E}}$, its energy in $(\vec{G}, w)$ is $\sum_{e \in \vec{E}} \frac{1}{w_e} f_e^2 =: \langle f, f \rangle_w$

Cycle and cut spaces (weighted extension)

Inner product

$\langle f, g\rangle_w := \sum_{e\in\vec{E}} \frac{1}{w_e} f_e g_e$

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Cycle space (same as before)

The cycle space of $(\vec{G}, w)$ is $W^\diamond := \{f : \nabla f = \mathbf{0}\}$.

Cut space (different)

The cut space of $(\vec{G}, w)$ is $\{g : \langle f, g\rangle_w = 0 \ \forall f \in W^\diamond\}$.

Cycle and cut spaces (weighted extension)

Inner product

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Cycle space

The cycle space of $(\vec{G}, w)$ is $W^\diamond := \{f : \nabla f = \mathbf{0}\}$.

Cut space

The cut space of $(\vec{G}, w)$ is $\{g : \langle f, g\rangle_w = 0 \ \forall f \in W^\diamond\}$.

Remark

$W^\diamond$ and the cut space are orthogonal complements with respect to the inner product $\langle\cdot,\cdot\rangle_w$.

Flow of minimum energy (weighted extension)

Let $b \in \mathbb{R}^V$ be a demands vector for which there is a flow f such that $b = \nabla f$.

Flow of minimum energy

$$\mathcal{E}_w(b) := \min \left\{ \langle g, g \rangle_w : \nabla g = b, g \in \mathbb{R}^{\vec{E}} \right\}$$

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Flow of minimum energy

$$\mathcal{E}_w(b) := \min\left\{\langle g, g\rangle_w : \nabla g = b, g \in \mathbb{R}^{\vec{E}}\right\}$$

Theorem

There is a unique flow of minimum energy, namely, the orthogonal projection of f onto the cut space of $(\vec{G}, w)$ with respect to the inner product $\langle \cdot, \cdot \rangle_w$.

Proof.

Exercise.

Rayleigh monotonicity principle (weighted extension)

Let $b \in \mathbb{R}^V$ be a demands vector for which there is a flow f such that $b = \nabla f$.

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Theorem

Consider $w, w' \in \mathbb{R}^{\vec{E}}_+$ such that $w \geq w'$. Then $\mathcal{E}_w(b) \leq \mathcal{E}_{w'}(b)$.

Proof.

Easy exercise.

Rayleigh monotonicity principle (weighted extension)

Let $b \in \mathbb{R}^V$ be a demands vector for which there is a flow f such that $b = \nabla f$.

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Theorem

Consider $w, w' \in \mathbb{R}_+^{\vec{E}}$ such that $w \geq w'$. Then $\mathcal{E}_w(b) \leq \mathcal{E}_{w'}(b)$.

Proof.

Easy exercise.

In particular,

- edge deletion increases effective resistance between two nodes,
- edge contraction decreases effective resistance between two nodes.