MA431 Spectral Graph Theory: Lecture 4

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7 The Laplacian matrix and spectrum

Let $G = (V, E)$ be a graph (recall that loops are not allowed by parallel edges are). Denote by $\Delta(G)$ the diagonal matrix corresponding to the vertex degrees of G. That is, the rows and columns of $\Delta(G)$ are indexed by V, and for each vertex $u \in V$, the uu-entry of $\Delta(G)$ is equal to $\deg(u)$. Recall that $A(G)$ is the adjacency matrix of G.

Definition 7.1. The *Laplacian matrix of* G is the real symmetric matrix $\Delta(G) - A(G)$.

An *orientation of* G is a directed graph D that is obtained from G by orienting every edge in an arbitrary direction. The *incidence matrix of* D is the $0, \pm 1$ matrix whose rows and columns are indexed by the vertices and arcs, respectively, where column (v, u) is equal to $e_u - e_v$.

Proposition 7.2. Let L be the Laplacian matrix of G. Then

1. $L = MM^\top$, where M is the incidence matrix of any orientation of G,
2. $L = \sum_{\{u,v\} \in E} (e_u - e_v)(e_u - e_v)^\top$,
3. for every $x \in \mathbb{R}^V$,
$$x^\top L x = \sum_{\{u,v\} \in E} (x_u - x_v)^2.$$

In particular, L is a positive semidefinite matrix.

Proof. Exercise. □

Definition 7.3. The *Laplacian spectrum of* G is the spectrum of its Laplacian matrix. If G has n vertices, then its spectrum is denoted $\lambda_1(G) \leq \cdots \leq \lambda_n(G)$.[1]

For general graphs, the Laplacian spectrum and the spectrum are not related; for example, it is possible for two cospectral graphs to have different Laplacian spectra (see Exercises). For regular graphs, however, the situation is different:

Theorem 7.4. Let G be an n-vertex graph that is k-regular. If G has spectrum $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$, then its Laplacian spectrum is $k - \theta_1 \leq k - \theta_2 \leq \ldots \leq k - \theta_n$.

[1] Note that for the Laplacian spectrum, λ_1 denotes the least eigenvalue, while for the usual spectrum, θ_1 denotes the largest eigenvalue.

Proof. Let $A := A(G)$, and let $v_1, \ldots, v_n$ be eigenvectors of A with eigenvalues $\theta_1, \ldots, \theta_n$, respectively. Let $L := \Delta(G) - A$ be the Laplacian matrix of G. As G is k-regular, $\Delta(G) = kI$, so $L = kI - A$. Subsequently,

$$Lv_i = (kI - A)v_i = (k - \theta_i)v_i,$$

implying in turn that $v_1, \ldots, v_n$ are also eigenvectors of L with eigenvalues $k - \theta_1, \ldots, k - \theta_n$, as claimed. □

Given that the Laplacian matrix is positive semidefinite, its eigenvalues are nonnegative. In fact, the least eigenvalue of the Laplacian spectrum is guaranteed to be 0:

Proposition 7.5. Let G be a graph with c connected components, let L be its Laplacian matrix, and let $\lambda_1 \leq \cdots \leq \lambda_n$ be the Laplacian spectrum. Then the following statements hold:

1. $L\mathbf{1} = \mathbf{0}$, that is, $\mathbf{1}$ is an eigenvector with eigenvalue 0. In particular, $\lambda_1 = 0$.
2. If $Lx = \mathbf{0}$, then x takes the same value on the vertices of each connected component of G.
3. $\operatorname{rank}(L) = n - c$. Equivalently, the eigenvalue 0 of L has multiplicity c.

Proof. **(1)** follows immediately from the definition of the Laplacian matrix.

(2) Let x be a vector such that $Lx = \mathbf{0}$. By Proposition 7.2,

$$0 = x^\top Lx = \sum_{\{u,v\} \in E} (x_u - x_v)^2,$$

implying that $x_u = x_v$ whenever u, v are adjacent. Thus x takes the same value on the vertices of each connected component, as required.

(3) Let $V_1, \ldots, V_c$ be the vertex sets of the connected components of G. For each $i \in [c]$, let $v_i \in \{0,1\}^V$ be the incidence vector of V_i. We claim that $v_1, \ldots, v_c$ is a basis for the null space of L, i.e. $\{x : Lx = \mathbf{0}\}$. Clearly, $Lv_i = \mathbf{0}$, and the v_i are linearly independent. Now choose a vector x such that $Lx = \mathbf{0}$. Then, by (2), x is a linear combination of $v_1, \ldots, v_c$. Thus, $v_1, \ldots, v_c$ is a basis for the null space of L, implying in turn that $\operatorname{rank}(L) = n - c$. □

Given that the least Laplacian eigenvalue is zero, one may ask questions about the second least Laplacian eigenvalue of a graph. Fiedler [2] calls $\lambda_2(G)$ the *algebraic connectivity of G*.

One can also get an upper-bound of n on the largest eigenvalue $\lambda_n(G)$ of the Laplacian of a simple graph G – see Exercises).

8 The Matrix-Tree Theorem

Let $G = (V, E)$ be an n-vertex graph, and let L be the Laplacian matrix. Denote by $T(G)$ the number of spanning trees of a graph – so if G is not connected, this number is zero. In this section, we prove Kirchhoff's

Matrix-Tree Theorem, which states that $T(G)$ is equal to the determinant of any $(n-1) \times (n-1)$ principal submatrix of L.

The Matrix-Tree Theorem is by and large a consequence of the Laplace (cofactor) expansion for the determinant, combined with a powerful *deletion-contraction* recursive formula for $T(G)$. To elaborate on the latter, let G be a graph, and let e be an edge. The *deletion* $G \setminus e$ is the graph obtained from G after removing the edge e. The *contraction* G/e is the graph obtained after identifying the ends of G, and deleting all the loops created.[2] Observe that contracting may create (additional) parallel edges.

Lemma 8.1. Let G be a graph. Then for every edge e,

$$T(G) = T(G/e) + T(G \setminus e).$$

Proof. The spanning trees of G can be separated into two groups, those that contain the edge e, and those that do not. The ones in the second group are precisely the spanning trees of $G \setminus e$. The ones are in the first group, however, are in correspondence with the spanning trees of G/e. More precisely, if T' is a spanning tree of G/e then $T' \cup \{e\}$ is a spanning tree of G containing e, and if T is a spanning tree of G containing e then $T - \{e\}$ is a spanning tree of $G \setminus e$. The formula above is an immediate consequence of this grouping of the spanning trees of G. □

In the next lecture, we will prove the following theorem:

Theorem 8.2 (Matrix-Tree Theorem)**.** Let G be an n-vertex graph, and let L be its Laplacian matrix. Then $T(G)$ is equal to the determinant of any $(n-1) \times (n-1)$ principal submatrix of L.

Acknowledgements

The presentation of §7 and §8 is inspired by [3], Chapter 13.

The Matrix-Tree Theorem dates back to the 1800s. Gustav Kirchhoff proved the "dual" of it in 1847 [4], but it was James Maxwell who stated the result explicitly in *A Treatise on Electricity and Magnetism, I* [5] (see Part II, Chapter 6, pp. 329-337). The theorem, as is, was stated and proved by Trent [6]. See also [1] for other references.

References

[1] S. Chaiken and D. Kleitman. Matrix tree theorems. *Journal of Combinatorial Theory, Series A*, 24:377–381, 1978.

[2] M. Fiedler. A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory. *Czechoslovak Mathematical Journal*, 25(4):619–633, 1975.

[3] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, New York, NY, 2000.

[2] In general, loops are not deleted after edge contractions, but in our context we must.

[4] G. Kirchhoff. Uber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer ströme gefuhrt wird. *Ann. Phys. Chem.*, 72(497-508), 1847.

[5] J. C. Maxwell. *A Treatise on Electricity and Magnetism, I*. Oxford University Press (Clarendon), London, 3rd ed. edition, 1892.

[6] H. M. Trent. Note on the enumeration and listing of all possible trees in a connected linear graph. *Proc. Nat. Acad. Sci. U.S.A.*, 40:1004–1007, 1954.