MA431 Assignment 1

Due Wednesday, Feb 2 at 10:00am GMT

Collect as many points as you can. The threshold for 100% is 35 points.

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

1. (Recommended, 10 points)

   Prove the Courant-Fischer Theorem (Theorem 0.2) parts (1), (3) and (5). Then apply those parts to $-A$ to prove parts (2), (4) and (6).

2. (Recommended, 5 points)

   Let G be a connected graph, and let $A := A(G)$. Prove that the following statements are equivalent:

   (a) G is bipartite,

   (b) the spectrum of G is symmetric about the origin, that is, if θ belongs to the spectrum, then so does $-\theta$, and both eigenvalues have the same multiplicity,

   (c) $-\rho(A)$ is an eigenvalue.

3. (Recommended, 5 points)

   Let $G = (V, E)$ be a graph, and let $A := A(G)$. Prove that

   $$\frac{\sum_{v \in V} \deg(v)}{|V|} \leq \rho(A) \leq \max\{\deg(v) : v \in V\},$$

   that is, $\rho(A)$ is sandwiched between the average degree and the maximum degree of G.

4. (Optional, 5 points)

   Find the spectrum of the Petersen graph. Show your work. (You may use a solver.)

5. (Optional, 5 points)

   Let G be a k-regular graph on n vertices with no loops or parallel edges, and let $k, \theta_2, \ldots, \theta_n$ be its spectrum. Prove that G and its complement $\overline{G}$ have a common set $v_1, v_2, \ldots, v_n$ of eigenvectors, which form a basis of $\mathbb{R}^n$, and that the eigenvalues of $\overline{G}$ are $n-1-k, -1-\theta_2, \ldots, -1-\theta_n$.

6. (Optional, 5 points)

   Let G be a connected graph with maximum degree Δ, and let $A = A(G)$. Prove that $\rho(A) = \Delta$ if, and only if, G is a Δ-regular graph.

7. (Recommended, 5 points)

   (Theorem 4.3) Let G be a graph, $A := A(G)$, and $\rho := \rho(A)$. Prove that G has chromatic number at most $1 + \lfloor \rho \rfloor$.

8. (Recommended, 5 points)

   Use Cauchy's Interlacing Theorem to prove that the Petersen graph has no Hamilton cycle.
   (You may use a solver to compute a graph spectrum.
   Hint. Consider the line graph of the Petersen.)