Lecture 4: Blocking and antiblocking theory

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1 The set covering and set packing problems

Let $\mathcal{C}$ be a family of subsets of a finite ground set V. What is the minimum number of elements needed to intersect every member of $\mathcal{C}$ at least once? This is known as *hitting set* problem, includes the *vertex cover* problem, and is equivalent to the *set covering problem*, and together form three of Karp's 21 NP-complete problems. We can model the hitting set problem by a nice integer program. Let A be the $0,1$ matrix whose columns are labeled by V, and whose rows are the incidence vectors of the members of $\mathcal{C}$. The problem can then be modelled as the following integer program for $w = \mathbf{1}$:

$$\textit{(set covering integer program)}\qquad \begin{array}{lrl} \min & w^\top x & \\ \text{s.t.} & Ax & \geq \mathbf{1} \\ & x & \geq \mathbf{0} \quad \text{integer} \end{array}$$

In general, for weights $w \in \mathbb{Z}^V$, this integer program finds a *minimum weight* set of elements intersecting every member at least once. Though the integer program above is NP-hard to solve, its linear programming relaxation can be solved efficiently:

$$\textit{(set covering linear program)}\qquad \begin{array}{lrl} \min & w^\top x & \\ \text{s.t.} & Ax & \geq \mathbf{1} \\ & x & \geq \mathbf{0} \end{array}$$

A natural question to ask now is, when is linear relaxation equivalent to the integer program, regardless of the choice of weights? In this case, the hitting set problem can be efficiently solvable.

Definition 1.1. A $0,1$ matrix A is *ideal* if the set covering linear program has an integral optimal solution for all $w \in \mathbb{Z}^V$. Equivalently, A is ideal if the polyhedron $\{x \in \mathbb{R}^n : Ax \geq \mathbf{1}, x \geq \mathbf{0}\}$ is integral.

Let us now ask another question. What is the maximum number of elements that intersect every member of $\mathcal{C}$ *at most* once? This includes the *independent set problem*, and is equivalent to the *set packing problem*, together forming two of Karp's 21 NP-complete problems. We can model the set packing problem by the following integer program for $w = \mathbf{1}$:

$$\textit{(set packing integer program)}\qquad \begin{array}{lrl} \max & w^\top x & \\ \text{s.t.} & Ax & \leq \mathbf{1} \\ & x & \geq \mathbf{0} \quad \text{integer} \end{array}$$

In general, for weights $w \in \mathbb{Z}^V$, this integer program finds a subset of *maximum weight* set intersecting every member of $\mathcal{C}$ at most once. Similarly as above, we can take the linear relaxation:

$$\text{(set packing linear program)} \qquad \begin{array}{lrl} \max & w^\top x & \\ \text{s.t.} & Ax & \leq \mathbf{1} \\ & x & \geq \mathbf{0} \end{array}$$

which, in general, is efficiently solvable unlike the set packing integer program. Once again, we may ask when the set packing linear program is equivalent to the integer program regardless of the choice of weights? for in this case, the set packing problem would be efficiently solvable. This leads to the following notion.

Definition 1.2. A $0,1$ matrix A without a zero column is *perfect* if the set packing linear program has an integral optimal solution for all $w \in \mathbb{Z}^V$. Equivalently, A is perfect if the polyhedron $\{x \in \mathbb{R}^n : Ax \leq \mathbf{1}, x \geq \mathbf{0}\}$ is integral.

In this lecture, we study ideal and perfect matrices. In particular, we shall see how T-cuts from lecture 1, and odd circuits of planar signed graphs from lecture 3, lead to ideal matrices, providing yet another motivation to study such matrices. We shall also see how perfect matrices from lecture 2 lead to perfect graphs, and surprisingly, vice versa!

2 Blocking and antiblocking polyhedra

Blocking theory. A polyhedron P is of *blocking type* if it is of the form $\{x \in \mathbb{R}^n : Ax \geq \mathbf{1}, x \geq \mathbf{0}\}$ where A is a nonnegative matrix. The *blocker of* P is

$$b(P) := \left\{z \in \mathbb{R}^n_+ : x^\top z \geq 1 \ \forall x \in P\right\}.$$

In the first part of this section, we see that $b(P)$ is also a polyhedron of blocking type, one whose blocker is P itself. We also see that every vertex of $b(P)$ is a row of A, and that if $Ax \geq \mathbf{1}$ includes no redundant inequality, then every row of A is a vertex of $b(P)$.

Proposition 2.1. Given a polyhedron $P \subseteq \mathbb{R}^n$ of blocking type, the following statements hold:

1. $b(P) = \{z \in \mathbb{R}^n_+ : Bz \geq \mathbf{1}, z \geq \mathbf{0}\}$, where the rows of B are the (finitely many) vertices of P,
2. $b(P)$ is of blocking type,
3. $b(b(P)) = P$.

Proof. **(1)** As P is of blocking type, its recession cone is $\mathbb{R}^n_+$, so every point in P is greater than or equal to a convex combination of the vertices. Thus, it suffices to check the infinitely many inequalities $x^\top z \geq 1 \ \forall x \in P$ only for the vertices x of P. Subsequently, a point z belongs to $b(P)$ if, and only if, $z \geq \mathbf{0}$ and $Bz \geq \mathbf{1}$, so $b(P)$ is polyhedral with description $b(P) = \{z \in \mathbb{R}^n_+ : Bz \geq \mathbf{1}, z \geq \mathbf{0}\}$. **(2)** As every vertex of P is nonnegative,

B is a nonnegative matrix, so $b(P)$ is of blocking type. **(3)** We may therefore speak of $b(b(P))$. It follows from definition that $P \subseteq b(b(P))$. To prove equality, pick a point $x^\star \notin P$. If $x^\star$ has a strictly negative entry, then $x^\star \notin b(b(P))$. Otherwise, $x^\star \geq \mathbf{0}$. As P is of blocking type, it has a facet-defining inequality $a^\top x \geq 1$ with $a \geq \mathbf{0}$, that is violated by $x^\star$, i.e. $a^\top x^\star < 1$. Clearly, $a \in b(P)$, so $a^\top x^\star < 1$ implies that $x^\star \notin b(b(P))$, as required. $\square$

We may therefore refer to $P, b(P)$ as a *blocking pair of polyhedra*. For example, the following polyhedron is of blocking type:

$$\left\{ x \geq \mathbf{0} : \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} x \geq \mathbf{1} \right\}$$

Its vertices are $(\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}), (1\ 1\ 0), (1\ 0\ 1), (0\ 1\ 1)$, so by Proposition 2.1, the blocker is

$$\left\{ z \geq \mathbf{0} : \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} z \geq \mathbf{1} \right\}.$$

Proposition 2.2. Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq \mathbf{1}, x \geq \mathbf{0}\}$ where A is a nonnegative matrix. Then every vertex of $b(P)$ is a row of A. Moreover, if every row of $Ax \geq \mathbf{1}$ is irredundant, then every row of A is also a vertex of $b(P)$.

Proof. Let a be a vertex of $b(P)$. Then $a^\top x \geq 1$ is a valid inequality for P, so it is implied by $Ax \geq \mathbf{1}, x \geq \mathbf{0}$, implying in turn that there exists $y \geq \mathbf{0}$ such that $a^\top \geq y^\top A$ and $y^\top \mathbf{1} \geq 1$. In fact, as A is a nonnegative matrix, we may assume that $y^\top \mathbf{1} = 1$. In particular, a is greater than or equal to a convex combination of the rows of A, each of which is clearly a point in $b(P)$. This, combined with the fact that a is a vertex of $b(P)$, implies that a must be a row of A. This proves the first statement.

For the second statement, pick a row a' of A that is not a vertex of $b(P)$, if any. The point a' clearly belongs to $b(P)$, so it must be greater than or equal to a convex combination of the vertices of $b(P)$, each of which is a row of A different from a'. Subsequently, the inequality $a'^\top x \geq 1$ can be dropped from $Ax \geq \mathbf{1}, x \geq \mathbf{0}$ without changing P, meaning that $a'^\top x \geq 1$ is redundant. This proves the second statement. $\square$

Theorem 2.3. Let A be a nonnegative matrix, and let B be the matrix whose rows are the vertices of $\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$. If A is a $0,1$ ideal matrix, then so is B.

Proof. Suppose A is a $0,1$ ideal matrix, that is, the set covering polyhedron $P := \{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ is integral. So B is a $0,1$ matrix. By Proposition 2.1, $b(P) = \{y \geq \mathbf{0} : By \geq \mathbf{1}\}$. Therefore, by Proposition 2.2, every vertex of $\{y \geq \mathbf{0} : By \geq \mathbf{1}\}$ is a row of A. In particular, $\{y \geq \mathbf{0} : By \geq \mathbf{1}\}$ is integral, that is, B is ideal. $\square$

Antiblocking theory. In the second part of this section, let us develop antiblocking theory of polyhedra, which is somewhat similar to blocking theory but with some subtle differences.

A polytope P is of *antiblocking type* if it is of the form $\{x \in \mathbb{R}^n : Ax \leq \mathbf{1}, x \geq \mathbf{0}\}$ where A is a nonnegative matrix without a zero column – to ensure that P is a bounded polyhedron. Observe that for a sufficiently small $\epsilon > 0$, $\epsilon\mathbf{1} \in P$. The *antiblocker of* P is the set

$$a(P) := \left\{z \in \mathbb{R}^n : x^\top z \leq 1 \ \forall x \in P, z \geq \mathbf{0}\right\}.$$

We shall see that $a(P)$ is a polytope of antiblocking type, and that its antiblocker is P itself. In contrast to blocking polyhedra, however, we can no longer claim that every vertex of $a(P)$ is a row of A. For example, $\mathbf{0}$ is always a vertex of $a(P)$, but is not necessarily a row of A. However, we will be able to salvage this and show that every vertex of $a(P)$ is a "orthogonal projection" of a row of A.

Proposition 2.4. Given a polytope $P \subseteq \mathbb{R}^n$ of antiblocking type, the following statements hold:

1. $a(P) = \{z \in \mathbb{R}^n : Bz \leq \mathbf{1}, z \geq \mathbf{0}\}$, where the rows of B are the (finitely many) vertices of P,
2. $a(P)$ is of antiblocking type,
3. $a(a(P)) = P$.

Proof. **(1)** Every point in P is a convex combination of its vertices. Thus, it suffices to check the infinitely many inequalities $x^\top z \leq 1 \ \forall x \in P$ only for the vertices x of P. Subsequently, a point z belongs to $a(P)$ if, and only if, $z \geq \mathbf{0}$ and $Bz \geq \mathbf{1}$, so $a(P)$ is polyhedral with description $a(P) = \{z \in \mathbb{R}^n : Bz \leq \mathbf{1}, z \geq \mathbf{0}\}$. **(2)** As every vertex of P is nonnegative, B is a nonnegative matrix. Moreover, since $\epsilon\mathbf{1} \in P$ for sufficiently small $\epsilon > 0$, it follows that B has no zero column, implying in turn that $a(P)$ is of antiblocking type. **(3)** We may therefore speak of $a(a(P))$. It follows from definition that $P \subseteq a(a(P))$. To prove equality, pick a point $x^\star \notin P$. If $x^\star$ has a strictly negative entry, then $x^\star \notin a(a(P))$. Otherwise, $x^\star \geq \mathbf{0}$. As P is of antiblocking type, it has a facet-defining inequality $a^\top x \leq 1$, where $a \geq \mathbf{0}$, that is violated by $x^\star$, i.e. $a^\top x^\star > 1$. Clearly, $a \in a(P)$, so $a^\top x^\star > 1$ implies that $x^\star \notin a(a(P))$, as required. $\square$

We may therefore refer to $P, a(P)$ as an *antiblocking pair of polytopes*. For example, consider the following polytope of antiblocking type:

$$\left\{x \geq \mathbf{0} : \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} x \leq \mathbf{1}\right\}$$

The vertices of this polytope are $(\frac{1}{2} \ \frac{1}{2} \ \frac{1}{2}), (1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1), (0\ 0\ 0)$, so by Proposition 2.4, the antiblocker is

$$\left\{z \geq \mathbf{0} : \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} z \leq \mathbf{1}\right\}$$

The vertices of the second polytope are $(1\ 1\ 0), (1\ 0\ 1), (0\ 1\ 1), (1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1), (0\ 0\ 0)$, each of which is either a row or an "orthogonal projection" of a row of the constraint matrix of the first polytope. Below, we show that this is true in general.

Given vectors x, y of the same dimension, if x is obtained from y after setting some of the coordinates to 0, then we say that x is a *projection* of y.

Lemma 2.5. Let A be a nonnegative matrix, and let $P := \{x \in \mathbb{R}^n : Ax \leq \mathbf{1}, x \geq \mathbf{0}\}$. Let $\bar{x}$ be a vertex of P for which

$$\bar{x} \leq \sum_{i=1}^{k} \lambda_i x^i$$

for some points $x^1, \ldots, x^k \in P$ and scalars $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$. Then $\bar{x}$ is a projection of each x^i.

Proof. If $\bar{x} = \mathbf{0}$, then we are done. Otherwise, after possibly rearranging the coordinates, we have $\bar{x} = (\bar{z}, \mathbf{0})$ for some $\ell \geq 1$ and $\bar{z} \in \mathbb{R}^\ell$ such that $\bar{z} > \mathbf{0}$. For each $i \in [k]$, denote by z^i the vector consisting of the first ℓ coordinates of x^i. Then

$$\bar{z} \leq \sum_{i=1}^{k} \lambda_i z^i =: z.$$

Notice that z consists of the first ℓ coordinates of $\sum_{i=1}^k \lambda_i x^i$. As $\bar{x}$ is an extreme point of P, there is an $\ell \times \ell$ nonsingular submatrix E of A such that $E\bar{z} = \mathbf{1}$. On the one hand, as E is nonnegative and $z \geq \bar{z}$, we have that $Ez \geq E\bar{z} = \mathbf{1}$. On the other hand, as $A\left(\sum_{i=1}^k \lambda_i x^i\right) \leq \mathbf{1}$, we have that $Ez \leq \mathbf{1}$. Thus, $Ez = E\bar{z} = \mathbf{1}$, implying in turn that $z = \bar{z}$. As a result,

$$\bar{x} = (\bar{z}, \mathbf{0}) = (z, \mathbf{0}) = \sum_{i=1}^{k} \lambda_i (z^i, \mathbf{0}).$$

Since $\bar{x}$ is an extreme point, and each $(z^i, \mathbf{0})$ belongs to P, it follows that $\bar{x} = (z^1, \mathbf{0}) = \cdots = (z^k, \mathbf{0})$, as required. $\square$

We are now ready to prove the following.

Proposition 2.6. Consider a polytope $P = \{x \in \mathbb{R}^n : Ax \leq \mathbf{1}, x \geq \mathbf{0}\}$ where A is a nonnegative matrix without a zero column. Then every vertex of $a(P)$ is a projection of some row of A.

Proof. Let a be a vertex of $a(P)$. Then $a^\top x \leq 1$ is a valid inequality for P, so it must be implied by $Ax \leq \mathbf{1}, x \geq \mathbf{0}$, implying in turn that there exists $y \geq \mathbf{0}$ such that $a^\top \leq y^\top A$ and $y^\top \mathbf{1} \leq 1$. In fact, as A is a nonnegative matrix, we may assume that $y^\top \mathbf{1} = 1$. In particular, a is less than or equal to a convex combination of the rows of A, each of which is clearly a point in $a(P)$. It then follows from Lemma 2.5 that a is a projection of each row used in the convex combination, thereby proving the result. $\square$

Consequently, we get the following analogue of the (Weak) Perfect Graph Theorem for perfect matrices:

Theorem 2.7 (Pluperfect Graph Theorem)**.** Let A be a nonnegative matrix without a column of all zeros, and let B be the matrix whose rows are the vertices of $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$. If A is a $0,1$ perfect matrix, then so is B.

Proof. Suppose A is a $0,1$ perfect matrix, that is, the set packing polytope $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral. Then B is a $0,1$ matrix. By Proposition 2.4, B has no zero column, and $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Therefore, by Proposition 2.6, every vertex of $\{y \geq \mathbf{0} : By \leq \mathbf{1}\}$ is a projection of a row of A. In particular, $\{y \geq \mathbf{0} : By \leq \mathbf{1}\}$ is integral, that is, B is perfect. $\square$

3 Clutters, the blocker, and the width-length inequality

In the previous section we developed blocking theory of polyhedra, culminating in Theorem 2.3. This theorem indicates that it is worthwhile to focus on $0,1$ matrices, which can be viewed as set systems, or "clutters". In this section, we develop blocking theory of clutters.

Let E be a finite set, and let $\mathcal{C}$ be a family of subsets of E called *members*. We say that $\mathcal{C}$ is a *clutter* over *ground set* E if no member contains another one.

Remark 3.1. Let $\mathcal{C}_1, \mathcal{C}_2$ be clutters over the same ground set, where every member of $\mathcal{C}_1$ contains a member of $\mathcal{C}_2$, and every member of $\mathcal{C}_2$ contains a member of $\mathcal{C}_1$. Then $\mathcal{C}_1 = \mathcal{C}_2$.

Proof. Take $C_1 \in \mathcal{C}_1$. Then C_1 contains a member C of $\mathcal{C}_2$, and C contains a member of $\mathcal{C}_1$. As $\mathcal{C}_1$ is a clutter, it must be that $C_1 \subseteq C \subseteq C_1$, implying in turn that $C = C_1$. Thus, $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Similarly, $\mathcal{C}_2 \subseteq \mathcal{C}_1$, so $\mathcal{C}_1 = \mathcal{C}_2$. $\square$

A *cover* of $\mathcal{C}$ is a subset of E that intersects every member at least once. A cover is *minimal* if it does not properly contain another cover. The family of minimal covers of $\mathcal{C}$ forms another clutter over the same ground set called the *blocker of* $\mathcal{C}$, and denoted $b(\mathcal{C})$.

Theorem 3.2. Given a clutter $\mathcal{C}$, we have $b(b(\mathcal{C})) = \mathcal{C}$.

Proof. Denote by E the ground set of $\mathcal{C}$. We need to show that the minimal covers of $b(\mathcal{C})$ are precisely the members of $\mathcal{C}$. By Remark 3.1, it suffices to show that (a) every member of $\mathcal{C}$ is a cover of $b(\mathcal{C})$, and (b) every minimal cover of $b(\mathcal{C})$ contains a member of $\mathcal{C}$.

a. Take $C \in \mathcal{C}$. Since $C \cap B \neq \emptyset$ for every $B \in b(\mathcal{C})$, we get that C is a cover of $b(\mathcal{C})$.

b. Take a minimal cover C' of $b(\mathcal{C})$. Then $E - C'$ cannot contain a member of $b(\mathcal{C})$, so $E - C'$ is not a cover of $\mathcal{C}$, implying in turn that $E - C'$ is disjoint from a member of $\mathcal{C}$. Consequently, C' contains a member of $\mathcal{C}$.

Thus, $b(b(\mathcal{C})) = \mathcal{C}$. $\square$

We can therefore speak of a blocking *pair* of clutters. Let us see some examples from previous lectures.

Let $G = (V, E)$ be a graph, and let T be a nonempty subset of V of even cardinality. Recall from lecture 1 that a T-join is an edge subset whose odd-degree vertices coincide with T, while a T-cut is a cut of the form $\delta(U) \subseteq E$ where $|U \cap T|$ is odd. It can be proved that the clutter of (inclusionwise) minimal T-joins and the clutter of minimal T-cuts form a blocking pair. We leave this as an exercise.

For another example, let $\Sigma \subseteq E$. Recall from lecture 3 that (G, Σ) is called a signed graph, that an odd circuit C of the signed graph is a circuit C such that $|C \cap \Sigma|$ is odd, and a signature is a set of the form $\Sigma \triangle \delta(U)$ for some $U \subseteq V$. It can be proved that the clutter of odd circuits and the clutter of minimal signatures form a blocking pair. We also leave this as an exercise.

Let $\mathcal{C}$ be a clutter over ground set E. The *incidence matrix* of $\mathcal{C}$ is the matrix whose columns are labeled by E, and whose rows are the incidence vectors of the members.

Lemma 3.3. Let $\mathcal{C}$ be a clutter, let A be its incidence matrix, and let $P = \{x \in \mathbb{R}^n : Ax \geq \mathbf{1}, x \geq \mathbf{0}\}$. Then the following statements are equivalent for x:

i. x is an integral vertex of P,

ii. x is the incidence vector of a minimal cover of $\mathcal{C}$.

Proof. Exercise. □

A clutter is *ideal* if its incidence matrix is an ideal matrix. For example, by the Edmonds-Johnson Theorem, the clutter of minimal T-cuts is an ideal clutter. Moreover, as we showed in lecture 3, the clutter of odd circuits of a planar signed graph is also ideal.

Theorem 3.4. A clutter is ideal if, and only if, its blocker is ideal.

Proof. Let $\mathcal{C}$ be an ideal clutter, let A be its incidence matrix which is ideal, and let $P = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$. Let B be the matrix whose rows are the vertices of P. As A is an ideal matrix, it follows from Theorem 2.3 that B is an ideal matrix. Moreover, as A is ideal, P has only integral vertices, so by Lemma 3.3, B is just the incidence matrix of $b(\mathcal{C})$. As a result, $b(\mathcal{C})$ is an ideal clutter, as required. □

Theorem 3.4 suggests that there should be a blocker-invariant characterisation of idealness. This is indeed the case.

Theorem 3.5 (The Width-Length Inequality)**.** Let $\mathcal{C}$ be a clutter over ground set E. Then $\mathcal{C}$ is ideal if, and only if, for all $w, \ell \in \mathbb{R}^E_+$,

$$\min\{w(C) : C \in \mathcal{C}\} \cdot \min\{\ell(B) : B \in b(\mathcal{C})\} \leq w^\top \ell.$$

Proof. Suppose first that $\mathcal{C}$ is ideal. Take $w, \ell \in \mathbb{R}^E_+$. Consider the following primal-dual pair of linear programs:

$$(P) \quad \begin{array}{lll} \min & \ell^\top x & \\ \text{s.t.} & x(C) & \geq 1 \quad \forall C \in \mathcal{C} \\ & x_e & \geq 0 \quad \forall e \in E \end{array} \qquad\qquad (D) \quad \begin{array}{lll} \max & \sum (y_C : C \in \mathcal{C}) & \\ \text{s.t.} & \sum (y_C : e \in C \in \mathcal{C}) & \leq \ell_e \quad \forall e \in E \\ & y_C & \geq 0 \quad \forall C \in \mathcal{C} \end{array}$$

As $\mathcal{C}$ is ideal, the set covering program (P) has an integral optimal solution, and the optimal value is equal to $\tau := \min\left\{\ell(B) : B \in b(\mathcal{C})\right\}$. By Strong LP Duality, there exists $y \in \mathbb{R}^{\mathcal{C}}_+$ such that

$$\begin{aligned} \sum \left(y_C : C \in \mathcal{C}\right) &= \tau \\ \sum \left(y_C : e \in C \in \mathcal{C}\right) &\leq \ell_e \quad \forall e \in E. \end{aligned}$$

Now we have

$$\begin{aligned}
w^\top \ell = \sum_{e\in E} w_e \ell_e \geq \sum_{e\in E} w_e \left[\sum \left(y_C : e \in C \in \mathcal{C}\right)\right] &= \sum_{C\in\mathcal{C}} y_C \cdot w(C) \\
&\geq \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \sum_{C\in\mathcal{C}} y_C \\
&= \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \tau \\
&= \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \min\left\{\ell(B) : B \in b(\mathcal{C})\right\},
\end{aligned}$$

as required. Suppose conversely that the width-length inequality holds for all $w, \ell \in \mathbb{R}^E_+$. We will show that $\mathcal{C}$ is ideal. To this end, take an arbitrary $\ell \in \mathbb{R}^E_+$, and let $x^\star$ be an optimal solution to (P). We will show that

$$\ell^\top x^\star = \min\left\{\ell(B) : B \in b(\mathcal{C})\right\},$$

thereby finishing the proof. Well, it is clear that $\leq$ holds above. We will prove that $\geq$ holds as well. By the width-length inequality,

$$\begin{aligned}
\ell^\top x^\star &\geq \min\left\{\ell(B) : B \in b(\mathcal{C})\right\} \cdot \min\left\{x^\star(C) : C \in \mathcal{C}\right\} \\
&\geq \min\left\{\ell(B) : B \in b(\mathcal{C})\right\}.
\end{aligned}$$

as required. □

The results of this section have the following consequence:

Corollary 3.6. The following statements hold:

1. The clutter of minimal T-joins of a graph is ideal.
2. The clutter of minimal signatures of a planar signed graph is ideal.

4 Perfect matrices to perfect graphs

In this section, we prove that for every perfect matrix, the maximal rows correspond to the maximal stable sets of a perfect graph. This is a consequence of two surprising results. The first one is that every perfect matrix comes from a graph:

Theorem 4.1. Let A be a perfect matrix whose column labels form the set V. Then the maximal rows of A are the incidence vectors of the maximal stable sets of a simple graph with vertex set V.

Proof. Let $G = (V, E)$ be the simple graph where distinct vertices u, v are *non*adjacent if and only if A has a row a such that $a_u = a_v = 1$.

Claim 1. Every row is the incidence vector of some stable set.

Proof of Claim. This follows immediately from construction. $\Diamond$

Claim 2. For every stable set S, there exists a row a such that $a_v = 1$ for all $v \in S$.

Proof of Claim. The claim holds for $|S| = 1$ as A has no zero column (by definition of a perfect matrix), and holds for $|S| = 2$ by construction. We may therefore assume that $|S| \geq 3$. Consider the set packing program $\max\left\{\sum_{v\in S} x_v : Ax \leq \mathbf{1},\ x \geq \mathbf{0}\right\}$. The optimal value of this program is clearly at least 1. Moreover, due to the construction, the integer optimal value of this program must be at most 1. As A is a perfect matrix, the optimal value of this program must also be at most, and therefore equal to, 1. This observation implies the claim: for otherwise, the vector $\bar{x} \in \mathbb{R}^V_+$ defined as

$$\bar{x}_v = \begin{cases} \frac{1}{|S|-1} & \text{if } v \in S; \\ 0 & \text{otherwise;} \end{cases}$$

would be a feasible solution with objective value $\sum_{v\in S} \bar{x}_v = \frac{|S|}{|S|-1} > 1$, a contradiction. $\Diamond$

Claims 1 and 2 prove the theorem. $\square$

In what follows, we aim to prove that the graph constructed above is perfect. To this end, let A be a perfect matrix whose column labels form the set V. Consider the following primal-dual pair of linear programs:

$$(P)\quad \begin{array}{lrl} \max & w^\top x & \\ \text{s.t.} & Ax & \leq \mathbf{1} \\ & x & \geq \mathbf{0} \end{array} \qquad\qquad (D)\quad \begin{array}{lrl} \min & \mathbf{1}^\top y & \\ \text{s.t.} & A^\top y & \geq w \\ & y & \geq \mathbf{0} \end{array} \qquad w \in \mathbb{Z}^V$$

As A is a perfect matrix, the set packing linear program (P) has an integral optimal solution, for all $w \in \mathbb{Z}^n$. What is surprising is that,

Theorem 4.2. Let A be a perfect matrix. Then (D) has an integral optimal solution for all $w \in \mathbb{Z}^V$.

Proof. Let $\omega \in \mathbb{Z}_+$ be the optimal value of (P). We prove by induction on $\omega \geq 0$ that (D) has an integral optimal solution. If $\omega = 0$, then as A has no zero column, it follows that $w \leq \mathbf{0}$, implying in turn that $\mathbf{0}$ is an optimal solution for (D). For the induction step, assume that $\omega \geq 1$. Take an arbitrary row a of A such that

$$a^\top x^\star = 1 \qquad \text{for all optimal solutions } x^\star \text{ of } (P).$$

(To find this row, take an optimal dual solution $\hat{y}$, and pick a so that $\hat{y}_a > 0$; apply Complementary Slackness.) We may assume that a is the first row of A. Consider the set packing primal-dual pair

$$(P')\quad \begin{array}{lrl} \max & (w-a)^\top x & \\ \text{s.t.} & Ax & \leq \mathbf{1} \\ & x & \geq \mathbf{0} \end{array} \qquad\qquad (D')\quad \begin{array}{lrl} \min & \mathbf{1}^\top y & \\ \text{s.t.} & A^\top y & \geq w-a \\ & y & \geq \mathbf{0} \end{array} \qquad c \in \mathbb{Z}^V$$

Clearly, the optimal value of (P') is at most ω, and our choice of a implies that it is exactly $\omega - 1$. Thus, by the induction hypothesis, (D') has an integral optimal solution $\bar{y} = (\bar{y}_1, \bar{z})$ of value $\omega - 1$. Let $y^\star := (\bar{y}_1 + 1, \bar{z})$. Then $y^\star$ is an integral feasible solution for (D) and has value ω, so it is optimal. This completes the induction step. $\square$

Recall from lecture 2 that a graph is perfect if in every induced subgraph, the chromatic number is equal to the clique number.

Theorem 4.3. Let $G = (V, E)$ be a graph, and let A be the incidence matrix of the maximal stable sets of G. If A is a perfect matrix, then G is a perfect graph.

Proof. Assume that A is a perfect matrix. Let $X \subseteq V$. It suffices to show that $\chi(G[X]) = \omega(G[X])$. To this end, let $w \in \{0,1\}^V$ be the incidence vector of X.

$$(P) \quad \begin{array}{lrl} \max & w^\top x & \\ \text{s.t.} & Ax & \le \mathbf{1} \\ & x & \ge \mathbf{0} \end{array} \qquad\qquad (D) \quad \begin{array}{lrl} \min & \mathbf{1}^\top y & \\ \text{s.t.} & A^\top y & \ge w \\ & y & \ge \mathbf{0} \end{array} \qquad w \in \mathbb{Z}^V$$

By definition, (P) has an integral optimal solution $x^\star$, and by Theorem 4.2, (D) has an integral optimal solution $y^\star$. By Strong Duality, $w^\top x^\star = \mathbf{1}^\top y^\star =: \omega$.

As A has no column of all zeros, every entry of $x^\star$ is either 0 or 1, so $x^\star$ is the incidence vector of a set $K \subseteq V$. As $Ax^\star \le \mathbf{1}$, K intersects every maximal stable set at most once, implying in turn that K is a clique (this can readily be checked by the reader). Moreover, $\omega = w^\top x^\star = |X \cap K|$.

As $y^\star$ is integral and has objective value ω, it yields a list of maximal stable sets $S_1, \ldots, S_\omega$, where some sets may be repeated. As $A^\top y^\star \ge w$, we have $X \subseteq S_1 \cup \ldots \cup S_\omega$.

Putting these together, we see that in $G[X]$, $X \cap K$ is an ω-clique, and $S_1 \cap X, \ldots, S_\omega \cap X$ are ω stable sets covering the vertices, implying in turn that $\omega(G[X]) = \chi(G[X]) = \omega$, as required. □

5 Perfect graphs to perfect matrices

In this section, we prove the converse of the result Theorem 4.3, that for every perfect graph, the incidence matrix of the stable sets forms a perfect matrix. This surprising result is a consequence of a key lemma about perfect graphs. To this end, let $G = (V, E)$ be a perfect graph, and take a vertex $v \in V$. To *duplicate* v is to introduce a new vertex $\bar{v}$, join it to all the neighbours of v, and then join it to $\bar{v}$.

Lemma 5.1. Duplication preserves perfection. That is, if G is a perfect graph, and G' is obtained from it after duplicating some vertex v, then G' is also perfect.

Proof. We need to show that for every induced subgraph of G', the maximum cardinality of a clique is equal to the chromatic number. Every induced subgraph of G' containing at most one of $v, \bar{v}$ is isomorphic to an induced subgraph of G, so equality holds for it by assumption. We may therefore focus on induced subgraphs H' containing both $v, \bar{v}$. We shall prove by induction on $|V(H')| \ge 2$ that $\chi(H') = \omega(H')$. This trivially holds for the base case $|V(H')| = 2$.

For the induction step, assume that $|V(H')| \ge 3$. Let $H := H' \setminus \bar{v}$. It can readily be checked that $\omega(H) \le \omega(H') \le \omega(H) + 1$ and $\chi(H) \le \chi(H') \le \chi(H) + 1$. As an induced subgraph of G, H is a perfect graph, so

$\chi(H) = \omega(H)$, implying in turn that $\omega(H) \leq \chi(H') \leq \omega(H) + 1$. In fact, the lower bound on $\chi(H')$ can be improved to

$$\omega(H') \leq \chi(H') \leq \omega(H) + 1.$$

Thus, if $\omega(H') = \omega(H) + 1$, then $\chi(H') = \omega(H')$, thereby completing the induction step.

Otherwise, $\omega(H') = \omega(H) =: \omega$. In particular, no ω-clique of H contains v, and in turn no ω-clique of H' contains either of $v, \bar{v}$. Take an optimal proper vertex-colouring of H. As the number of colour classes is ω, every ω-clique picks exactly one vertex from each colour class. Let $S \subseteq V(H)$ be the colour class containing v. As v avoids every ω-clique, $S' := S \setminus v$ intersects every ω-clique exactly once. As a consequence,

$$\omega(H' \setminus S') = \omega - 1.$$

$H' \setminus S'$ is a proper induced subgraph of H' containing both $v, \bar{v}$, so it can be properly vertex-coloured by $\omega - 1$ colours, by the induction hypothesis. By colouring all the vertices in the stable set S' with a new colour, one obtains a proper vertex-colouring of H' by ω colours. Thus, $\chi(H') = \omega = \omega(H')$, thereby completing the induction step. $\square$

We are now ready for the main result of this section:

Theorem 5.2. Let $G = (V, E)$ be a perfect graph. Then A, the matrix whose columns are labeled by V and whose rows are the incidence vectors of the stable sets of G, is a perfect matrix.

Proof. Consider the set packing primal-dual pair

$$(P)\quad \begin{array}{lrl} \max & w^\top x & \\ \text{s.t.} & Ax & \leq \mathbf{1} \\ & x & \geq \mathbf{0} \end{array} \qquad\qquad (D)\quad \begin{array}{lrl} \min & \mathbf{1}^\top y & \\ \text{s.t.} & A^\top y & \geq w \\ & y & \geq \mathbf{0} \end{array} \qquad w \in \mathbb{Z}^V$$

We need to show that (P) has an integer optimum, for all $w \in \mathbb{Z}^V$. If $w_v < 0$ for some $v \in V$, then we may set $x_v = 0$ without affecting the optimal value of (P). We may therefore assume that $w \in \mathbb{Z}^V_+$. Let us rewrite the primal

$$(P)\quad \begin{array}{lll} \max & \sum (w_v x_v : v \in V) & \\ \text{s.t.} & \sum (x_v : v \in S) \leq 1 & \forall \text{ stable sets } S \\ & x_v \geq 0 & \forall\, v \in V. \end{array}$$

Let $K \subseteq V$ be a clique of maximum weight $w(K) = \sum_{v \in K} w_v$, and let $x^\star \in \{0,1\}^V$ be its incidence vector. Clearly, $x^\star$ is an integer feasible solution to (P). We prove that $x^\star$ is in fact optimal by presenting a dual feasible solution of value $w^\top x^\star$.

To this end, let G_w be the graph obtained from G after replacing each vertex v by w_v duplicates. (If $w_v = 0$ then delete v.) Observe that the w_v duplicates form a clique of G_w by construction. It can readily be checked that maximum weight cliques in G correspond to maximum cardinality cliques in G_w, so that $w^\top x^\star = w(K) = \omega(G_w)$.

As G is a perfect graph, so is G_w by Lemma 5.1. Thus, $\omega(G_w) = \chi(G_w)$. Any optimal proper vertex-colouring of G_w is a covering of the vertices of G_w by $\chi(G_w)$ stable sets. As every stable set picks at most one of the w_v duplicates of every vertex v of G, the covering of G_w yields a covering of G by $\chi(G_w)$ stable sets where every vertex is covered at least w_v times. That is, the dual program

$$(D) \quad \begin{array}{lll} \min & \sum(y_S : \text{ stable sets } S) & \\ \text{s.t.} & \sum(y_S : \text{ stable sets } S \text{ such that } v \in S) \geq w_v & \forall\, v \in V \\ & y_S \geq 0 & \forall \text{ stable sets } S \end{array}$$

has an integer feasible solution $y^\star$ with objective value $\chi(G_w) = \omega(G_w) = w^\top x^\star$.

Thus, $w^\top x^\star$ is the common optimal value of (P) and (D), implying in turn that $x^\star$ (and $y^\star$) is an integer optimum to (P) (and (D)), as required. $\square$

Exercises

1. Let A be a $0,1$ matrix, and let $P = \{x \in \mathbb{R}^n : Ax \geq \mathbf{1}, x \geq \mathbf{0}\}$. Prove that the following statements are equivalent:

   i. every row of $Ax \geq \mathbf{1}$ is irredundant,

   ii. the rows of A are incomparable, i.e. A is the incidence matrix of a clutter.

2. Prove Lemma 3.3.

3. Let
$$A := \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

   (a) Prove that A is an ideal matrix.

   (b) Consider the dual of the set covering linear program associated with A:
$$(D) \quad \begin{array}{lrl} \max & \mathbf{1}^\top y & \\ \text{s.t.} & A^\top y & \leq \mathbf{1} \\ & y & \geq \mathbf{0} \end{array}$$
   Prove that (D) has no integral optimal solution.

4. Let $\mathcal{C}$ be a clutter over ground set E, and let $R \cup B$ be a partition of E. Prove that either R contains a member of $\mathcal{C}$, or B contains a member of $b(\mathcal{C})$, but not both.

5. Let $\mathcal{C}$ be a clutter over ground set E. Prove that for every member C and element $e \in C$, there is a minimal cover B such that $C \cap B = \{e\}$.

6. Let $G = (V, E)$ be a graph, and let s, t be distinct vertices. Let $\mathcal{C}$ be the clutter over ground set E whose members are the st-paths of G. Use Menger's Theorem to prove that $\mathcal{C}$ is an ideal clutter.

7. Let $G = (V, E)$ be a graph, and let s, t be distinct vertices. Let $\mathcal{C}$ be the clutter over ground set E whose members are the st-paths of G. Use the width-length inequality to prove that $\mathcal{C}$ is an ideal clutter.

8. Let $G = (V, E)$ be a graph, and let $\mathcal{C}$ be the clutter over ground set E whose members are the spanning trees of G. Give a description of $b(\mathcal{C})$.

9. Let $G = (V, E)$ be a graph, and let T be a nonempty subset of V of even cardinality. Recall from Lecture 1 that a T-join is an edge subset whose odd-degree vertices coincide with T, while a T-cut is a cut of the form $\delta(U) \subseteq E$ where $|U \cap T|$ is odd.

   Prove that the clutter of (inclusionwise) minimal T-joins and the clutter of minimal T-cuts form a blocking pair.

10. Let $G = (V, E)$ be a graph, and let $\Sigma \subseteq E$. Recall from lecture 3 that (G, Σ) is called a signed graph, that an odd circuit C of the signed graph is a circuit C such that $|C \cap \Sigma|$ is odd, and a signature is a set of the form $\Sigma \triangle \delta(U)$ for some $U \subseteq V$. Prove that the clutter of odd circuits and the clutter of minimal signatures form a blocking pair.

11. Let $\mathcal{C}$ be a clutter over ground set V, where every element in V is contained in at least one member. The *antiblocker* of a clutter $\mathcal{C}$, denoted $a(\mathcal{C})$, is the clutter over ground set V whose members are the inclusionwise maximal sets in $\{A \subseteq V : |A \cap C| \leq 1 \ \forall C \in \mathcal{C}\}$.

    (a) Give an example of a clutter $\mathcal{C}$ where $a(a(\mathcal{C})) \neq \mathcal{C}$.

    (b) Assume that $a(a(\mathcal{C})) = \mathcal{C}$. Prove that $\mathcal{C}$ is the clutter of maximal stable sets of a graph.

12. A clutter is *perfect* if its incidence matrix is perfect. Prove that a clutter is perfect if, and only if, its antiblocker is perfect.

13. Let $\mathcal{C}$ be a clutter over ground set V, where every element in V is contained in at least one member. Prove that the following statements are equivalent:

    (a) $\mathcal{C}$ is perfect,

    (b) for all $\ell, w \in \mathbb{R}^V_+$,
    $$\max\{\ell(C) : C \in \mathcal{C}\} \cdot \max\{w(A) : A \in a(\mathcal{C})\} \geq \ell^\top w.$$

14. Let $\mathcal{C}$ be a clutter over ground set V, where every element in V is contained in at least one member, and let $P = \{x \in \mathbb{R}^V : x(C) \leq 1 \ \forall C \in \mathcal{C}, x \geq \mathbf{0}\}$. Prove that $x^\star$ is an integral vertex of P if, and only if, $x^\star$ is the incidence vector of a subset of V that intersects every member of $\mathcal{C}$ at most once.

15. In lecture 2, we proved the Perfect Graph Theorem, stating that graph perfection is closed under complementation. Use the results of this lecture to give an alternate proof of this theorem.

16. Let P, Q be antiblocking polytopes in $\mathbb{R}^n_+$. Take $p \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n p_i = 1$. Prove that

$$\max_{a \in P} \sum_{i=1}^n p_i \log a_i + \max_{b \in Q} \sum_{i=1}^n p_i \log b_i = \sum_{i=1}^n p_i \log p_i.$$

17. Let P, Q be antiblocking polytopes in $\mathbb{R}^n_+$. Let $\alpha := \max\{x_1 \cdots x_n : x \in P\}$ and $\beta := \max\{y_1 \cdots y_n : y \in Q\}$. Prove that $\alpha\beta = \frac{1}{n^n}$.

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