CO 750 Packing and Covering: Lecture 6

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5 Perfect graphs

Let $G = (V, E)$ be a simple graph. Recall that $\chi(G)$ is the chromatic number of G, or equivalently, it is the minimum number of stable sets needed to cover V. Moreover, $\omega(G)$ is the maximum cardinality of a clique. Recall that

$$\chi(G) \geq \omega(G).$$

Last time, we proved that

Theorem 5.4. *$\chi(G) = \omega(G)$ if G is any of the following graphs:*

(1) G or $\overline{G}$ is bipartite,

(2) G or $\overline{G}$ is the line graph of a bipartite graph,

(3) G or $\overline{G}$ is a comparability graph.

However, equality does not always hold in $\chi \geq \omega$. For instance, for the odd cycle C_5 on five vertices, $\chi(C_5) = 3 > 2 = \omega(C_5)$. Can we characterize when equality does hold? Is this even a well-posed question? Let H be an arbitrary graph, and let $k := \chi(H) - \omega(H) \geq 0$. Let $C \subseteq V(H)$ be a maximum clique of H. Let G be the graph obtained from H after adding k vertices and just enough edges so as to grow C into a clique of cardinality $\omega(H) + k$. Notice now that $\chi(G) = \chi(H) = \omega(H) + k = \omega(G)$. Starting from an arbitrary graph, we just constructed a graph for which equality holds in $\chi \geq \omega$. This construction tells us that asking when equality holds in

$$\chi \geq \omega$$

is an ill-posed question. To make sure this construction is ruled out, we will come up with a stronger notion.

Let $G = (V, E)$ be a simple graph. For $X \subseteq V$, the subgraph of G induced on vertices X is called an *induced subgraph* and is denoted $G[X]$. We say that G is *perfect* if, for every induced subgraph G' of G, $\chi(G') = \omega(G')$. (Notice that G' may be G.) In words, a simple graph is perfect if in each induced subgraph, the maximum cardinality of a clique is equal to the chromatic number. It follows from the preceding theorem that,

Corollary 5.5. *The following graphs are perfect:*

(1) bipartite graphs, and their complements,

(2) line graphs of bipartite graphs, and their complements,

(3) comparability graphs, and their complements.

The obvious question this corollary leads to is, does complementation preserve perfection? Claude Berge asked the same question in 1961. Although this may seem too good to be true, the answer is yes!

5.1 The max-max inequality and the weak perfect graph theorem

As a tribute to Manfred Padberg, we follow Gasparyan (1996) for the proof of the following result:

Theorem 5.6 (Lovász 1972)**.** *Let G be a simple graph. The following statements are equivalent:*

(i) G is perfect,

(ii) $\omega(H) \cdot \alpha(H) \geq |V(H)|$ for every induced subgraph H.

Proof. **(i)** $\Rightarrow$ **(ii)**: Let H be an induced subgraph. By definition, $\chi(H) = \omega(H)$, that is, $V(H)$ can be covered by $\omega(H)$ stable sets. Since each stable set has cardinality at most $\alpha(H)$, it follows that

$$|V(H)| \leq \omega(H) \cdot \alpha(H).$$

(ii) $\Rightarrow$ **(i)**: Suppose for a contradiction that G is not perfect. Let H be an induced subgraph of G that is not perfect, but every proper induced subgraph of H is perfect. Let $\omega := \omega(H)$, $\alpha := \alpha(H)$ and $n := |V(H)|$. Note that $n > 1$. Clearly,

$$\omega \geq \omega(H \setminus S) \geq \omega - 1 \qquad \text{for every non-empty stable set } S \subseteq V(H);$$

since $H \setminus S$ is perfect and H is not, it follows that

$$\omega(H \setminus S) = \omega \qquad \text{for every non-empty stable set } S \subseteq V(H).$$

Let S_0 be a maximum stable set of H. Then for every vertex $v \in S_0$, $H \setminus v$ is perfect, so its vertices can be partitioned into $\omega(H \setminus v) = \omega$ non-empty stable sets. As S_0 has α vertices, we get $\alpha\omega$ stable sets $S_1, \ldots, S_{\alpha\omega}$.

Claim. *Every maximum clique of H intersects all but one of $S_0, S_1, \ldots, S_{\alpha\omega}$ exactly once.*

Proof of Claim. Let C be a maximum clique of H. Clearly C intersects each one of $S_0, S_1, \ldots, S_{\alpha\omega}$ at most once. For a vertex $v \in S_0$, if

- $v \in C$: then C intersects all but one stable set in every partition of $V(H \setminus v)$ into ω stable sets,
- $v \notin C$: then C intersects all stable sets in every partition of $V(H \setminus v)$ into ω stable sets.

This observation immediately implies the claim. $\Diamond$

For each $i \in \{0, 1, \ldots, \alpha\omega\}$, let C_i be a maximum clique of $H \setminus S_i$; notice that $|C_i| = \omega$. Let A be the $0-1$ matrix whose columns are labeled by $V(H)$, and whose rows are the incidence vectors of $S_0, S_1, \ldots, S_{\alpha\omega}$. Let B be the $0-1$ matrix whose columns are labeled by $V(H)$, and whose rows are the incidence vectors of $C_0, C_1, \ldots, C_{\alpha\omega}$. It then follows from the claim above that $AB^\top = J - I$, where J is the all-ones matrix and I the identity matrix of appropriate dimensions. Since $J - I$ is a non-singular $(\alpha\omega + 1) \times (\alpha\omega + 1)$ matrix, it follows that both A and B have full row rank, implying in turn that

$$|V(H)| = n \geq \alpha\omega + 1 = \alpha(H) \cdot \omega(H) + 1 > |V(H)|,$$

a contradiction. $\square$

As a consequence, we get the so-called weak perfect graph theorem:

Theorem 5.7 (Lovász 1972)**.** *If a graph is perfect, then so is its complement.*

Proof. Suppose that G is perfect. Then by Theorem 5.6, for every induced subgraph H of G,

$$\omega(H) \cdot \alpha(H) \geq |V(H)|,$$

implying in turn that for every induced subgraph $\overline{H}$ of $\overline{G}$,

$$\alpha(\overline{H}) \cdot \omega(\overline{H}) \geq |V(\overline{H})|,$$

so by Theorem 5.6, $\overline{G}$ is perfect, as required. $\square$

5.2 Odd holes and odd antiholes

We say that a simple graph is *minimally imperfect* if it is not perfect, but every proper induced subgraph is perfect. Equivalently, a simple graph G is minimally imperfect if $\chi(G) > \omega(G)$, but for every proper induced subgraph G', $\chi(G') = \omega(G')$. The latter implies that a minimally imperfect graph is always connected.

Remark 5.8. *A graph is perfect if, and only if, it has no minimally imperfect induced subgraph.*