CO 750 Packing and Covering: Lecture 23

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11 Cube-ideal sets

Take an integer $n \geq 1$. Recall that a set $S \subseteq \{0,1\}^n$ is cube-ideal if its convex hull $\mathrm{conv}(S)$ can be described by hypercube inequalities

$$1 \geq x_i \geq 0 \qquad i \in [n]$$

and generalized set covering inequalities:

$$\sum_{i \in I} x_i + \sum_{j \in J}(1 - x_j) \geq 1 \qquad \text{for disjoint } I, J \subseteq [n].$$

Recall further that the cuboid of S, denoted $\mathrm{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members have incidence vectors

$$(x_1, 1 - x_1, x_2, 1 - x_2, \ldots, x_n, 1 - x_n) \qquad x \in S.$$

Note that $\{2i-1, 2i\}, i \in [n]$ are covers of $\mathrm{cuboid}(S)$, and that every member of $\mathrm{cuboid}(S)$ has cardinality n. Today we will show a set is cube-ideal if and only if its cuboid is ideal.

11.1 Ideal cuboids

Let $\mathcal{C}$ be a clutter over ground set E. Denote by $Q(\mathcal{C})$ the set covering polyhedron

$$\left\{x \in \mathbb{R}^E_+ : x(C) \geq 1 \;\; C \in \mathcal{C}\right\}.$$

Here, $x(C) = \sum (x_e : e \in C)$. Two elements of a clutter are *coexclusive* if they are never used together in a minimal cover. We will need the following basic result on coexclusive elements:

Theorem 11.2 (Abdi, Cornuéjols, Pashkovich 2018)**.** *Let $\mathcal{C}$ be a clutter and take distinct elements e, f. The following statements are equivalent:*

(i) e, f are coexclusive,

(ii) for all members C_e, C_f such that $C_e \cap \{e,f\} = \{e\}$ and $C_f \cap \{e,f\} = \{f\}$, $(C_e \cup C_f) - \{e,f\}$ contains another member,

(iii) for every extreme point $x^\star$ of $Q(\mathcal{C})$, $x^\star_e + x^\star_f \leq 1$.

Proof. **(i) ⇒ (ii)**: Suppose e, f are coexclusive elements of clutter $\mathcal{C}$. Take members C_e, C_f where $C_e \cap \{e, f\} = \{e\}$ and $C_f \cap \{e, f\} = \{f\}$. We will show that $C_e \cup C_f - \{e, f\}$ contains a member, thereby proving (ii). Suppose otherwise. Then the complement of $C_e \cup C_f - \{e, f\}$ is a cover, so it contains a minimal cover B. Since $B \cap C_e \neq \emptyset$ and $B \cap C_f \neq \emptyset$, we get that $\{e, f\} \subseteq B$, contradicting the fact that e, f are coexclusive. **(ii) ⇒ (iii)**: Take an extreme point $x^\star$ of $Q(\mathcal{C})$. We will show that $x^\star_e + x^\star_f \leq 1$, proving (iii). If $x^\star_e = 0$ or $x^\star_f = 0$, then clearly $x^\star_e + x^\star_f \leq 1$. Otherwise, there is a member C_e with $e \in C_e$ and a member C_f with $f \in C_f$ such that $x^\star(C_e) = x^\star(C_f) = 1$. If $\{e, f\} \subseteq C_e$, then $x^\star_e + x^\star_f \leq x^\star(C_e) = 1$. We may therefore assume that $C_e \cap \{e, f\} = \{e\}$ and, similarly, $C_f \cap \{e, f\} = \{f\}$. It now follows from (ii) that there is a member $C \subseteq C_e \cup C_f - \{e, f\}$. Then

$$x^\star_e + x^\star_f + 1 \leq x^\star_e + x^\star_f + x^\star(C) \leq x^\star(C_e) + x^\star(C_f) = 2,$$

proving (iii). **(iii) ⇒ (i)**: Since the incidence vector of every minimal cover B is an extreme point $x^\star$ of $Q(\mathcal{C})$, we get from $x^\star_e + x^\star_f \leq 1$ that B contains at most one e, f. So e, f are coexclusive, proving (i). □

Recall that if $\mathcal{C}$ is ideal, then the extreme points of $Q(\mathcal{C})$ are precisely the incidence vectors of the minimal covers, so $Q(\mathcal{C}) = \mathrm{conv}\left(\{\chi_B : B \in b(\mathcal{C})\}\right) + \mathbb{R}^E_+$. We will need this below:

Lemma 11.3 (Guenin 1998, Nobili and Sassano 1998)**.** *Take a clutter $\mathcal{C}$ over ground set $E = \{e_1, f_1, \ldots, e_n, f_n\}$, where for each $i \in [n]$, $\{e_i, f_i\}$ intersects every member exactly once. Then the following statements are equivalent:*

(i) $b(\mathcal{C})$ is ideal,

(ii) $\mathrm{conv}\big\{\chi_C : C \in \mathcal{C}\big\} = Q\big(b(\mathcal{C})\big) \cap \big\{x : x_{e_i} + x_{f_i} = 1 \ \forall i \in [n]\big\}$.

Proof. **(i) ⇒ (ii)**: Since $\chi_C \in \big\{x : x_{e_i} + x_{f_i} = 1 \ \forall i \in [n]\big\}$ for every member C, the inclusion $\subseteq$ holds. Let us prove the reverse inclusion $\supseteq$. Since $b(\mathcal{C})$ is ideal, we get that

$$Q\big(b(\mathcal{C})\big) = \mathrm{conv}\big\{\chi_C : C \in \mathcal{C}\big\} + \mathbb{R}^E_+.$$

It is easy to see that this equation implies the reverse inclusion. **(ii) ⇒ (i)**: Let $x^\star$ be an extreme point of $Q\big(b(\mathcal{C})\big)$. It suffices to show that $x^\star$ is integral. Since $\{e_i, f_i\}$ is a cover of $\mathcal{C}$, it contains a member of $b(\mathcal{C})$, so $x^\star_{e_i} + x^\star_{f_i} \geq 1$. Moreover, since e_i, f_i are exclusive in $\mathcal{C}$, they are coexclusive in $b(\mathcal{C})$, so by Theorem 11.2 (iii), $x^\star_{e_i} + x^\star_{f_i} \leq 1$. So for each $i \in [n]$, $x^\star_{e_i} + x^\star_{f_i} = 1$, implying in turn by (ii) that $x^\star \in \mathrm{conv}\big\{\chi_C : C \in \mathcal{C}\big\}$. Since $x^\star$ is an extreme point, it must be one of the incidence vectors and hence integral, as required. □

We are now ready to prove the following:

Theorem 11.4 (Abdi, Cornuéjols, Guričanová, Lee 2018+)**.** *Take an integer $n \geq 1$ and a subset $S \subseteq \{0, 1\}^n$. Then S is cube-ideal if, and only if, $\mathrm{cuboid}(S)$ is an ideal clutter.*

Proof. Let $\mathcal{C} := \text{cuboid}(S)$. Notice that $\mathcal{C}$ is over ground set $E = \{1, 2, \ldots, 2n-1, 2n\}$, where for each $i \in [n]$, $\{2i-1, 2i\}$ intersects every member exactly once. We may therefore apply Lemma 11.3. ($\Leftarrow$) Assume that $\mathcal{C}$ is ideal. It follows from Theorem 7.8 that $b(\mathcal{C})$ is an ideal clutter also. Thus by Lemma 11.3, we have that

$$\text{conv}\{\chi_C : C \in \mathcal{C}\} = Q\big(b(\mathcal{C})\big) \cap \{x : x_{2i-1} + x_{2i} = 1 \;\forall i \in [n]\}.$$

Eliminating the even coordinates using the Fourier-Motzkin elimination method, we get that

$$\text{conv}(S) = \left\{ y \in [0,1]^n : \sum \big(y_i : 2i-1 \in B\big) + \sum \big(1 - y_j : 2j \in B\big) \geq 1 \quad \forall B \in b(\mathcal{C}) \right\}.$$

As a result, S is cube-ideal. ($\Rightarrow$) Assume conversely that S is cube-ideal, so

$$\text{conv}(S) = \left\{ y \in [0,1]^n : \sum \big(y_i : i \in I\big) + \sum \big(1 - y_j : j \in J\big) \geq 1 \quad \forall\, (I, J) \in \mathcal{V} \right\},$$

for some appropriate set $\mathcal{V}$. We may assume that for each $(I, J) \in \mathcal{V}$, $I \cap J = \emptyset$. After the change of variables $y_i \mapsto x_{2i-1}$ and $1 - y_i \mapsto x_{2i}$ to the equation above, we get that

$$\text{conv}\{\chi_C : C \in \mathcal{C}\} = \left\{ x \in \mathbb{R}^{2n}_+ : \begin{array}{l} \sum \big(x_{2i-1} : i \in I\big) + \sum \big(x_{2j} : j \in J\big) \geq 1 \quad \forall\, (I, J) \in \mathcal{V} \\ x_{2i-1} + x_{2i} = 1 \quad \forall\, i \in [n] \end{array} \right\}.$$

Together with Lemma 11.3, this equation implies that $b(\mathcal{C})$ is an ideal clutter, so by Theorem 7.8, $\mathcal{C}$ is an ideal clutter, as required. $\square$

11.2 The sums of circuits property

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. We say that S is a *binary space* (or a *vector space over $GF(2)$*) if

- $\mathbf{0} \in S$, and
- if $a, b \in S$ then $a \triangle b \in S$.

When is a binary space cube-ideal? To answer this question, we need to introduce some terminology. The *orthogonal complement of S* is

$$S^\perp := \left\{ d \in \{0,1\}^n : d^\top c \equiv 0 \pmod 2 \quad \forall c \in S \right\}.$$

It is clear that $S^\perp$ is another binary space, and it is widely known that $(S^\perp)^\perp = S$. To describe $S^\perp$ explicitly, we first write

$$S = \left\{ x \in \{0,1\}^n : Ax \equiv \mathbf{0} \pmod 2 \right\}$$

for some $m \times n$ matrix A with $0-1$ entries. Then $S^\perp$ is equal to the row space of A modulo 2:

$$S^\perp = \left\{ A^\top x : x \in \{0,1\}^m \right\}.$$

Denote by E the column labels of A. We say that a subset $C \subseteq E$ is a *cycle* if $\chi_C \in S$, and that a subset $D \subseteq E$ is a *cocycle* if $\chi_D \in S^\perp$. Notice that a cycle and a cocycle will always have an even number elements in common.

Example. *Let $G = (V, E)$ be a graph where loops are viewed as vertex-less edges. Then*

$$S := \{\chi_C : C \subseteq E \text{ is a graph cycle}\} \subseteq \{0,1\}^E$$

is a binary space, because for graph cycles C_1, C_2, their symmetric difference $C_1 \triangle C_2$ is also a graph cycle. We can represent S as

$$S = \left\{x \in \{0,1\}^E : Ax \equiv \mathbf{0} \pmod 2\right\}$$

where A is the vertex-edge incidence matrix of G. As a result, the cocycles of S correspond to the points in the row space of A modulo 2, implying in turn that the cocycles of S are precisely the cuts of G.