CO 750 Packing and Covering: Lecture 22

Ahmad Abdi

July 18, 2017

10.3 A signed graph without an odd-K_5 minor is weakly bipartite.

Recall that a pseudo-odd-K_5 is a signed graph $(G, E(G))$ for which the following statements hold: there exist a partition of $V(G)$ into parts S_0, S_1, S_2, S_3 and distinct vertices $x, y \in S_0$ such that

- there is an edge $e \in E$ whose ends are x, y, and for each $i \in \{0, 1, 2, 3\}$, S_i is stable in $G \setminus e$,
- $G \setminus e$ has internally vertex-disjoint xy-paths P_1, P_2, P_3, where for each $i \in [3]$, $V(P_i) \subseteq S_0 \cup S_i$,
- for distinct $i, j \in [3]$, $G[S_i \cup S_j]$ has a path with one end in $V(P_i)$ and the other in $V(P_j)$.

We showed last time that,

Theorem 10.9. *A pseudo-odd-K_5 has an odd-K_5 minor.*

Using this result, we will be able to continue with our proof of the following theorem. Our proof is due to Schrijver (2002).

Theorem 10.10 (Guenin 2001)**.** *A signed graph without an odd-K_5 minor is weakly bipartite.*

Proof. Let $(G = (V, E), \Sigma)$ be a signed graph that is not weakly bipartite. We will show that (G, Σ) has an odd-K_5 minor. To this end, as we argued last time, we may assume that G is connected, and that every proper minor of (G, Σ) is weakly bipartite. Let $\mathcal{C}$ be the clutter of odd circuits of (G, Σ). It then follows from Proposition 10.6 that $\mathcal{C}$ is a minimally non-ideal clutter. Take an edge $e \in E$. Using Lehman's Theorem 9.12, we showed the following last time:

Claim 1. *There are minimum odd circuits C_1, C_2, C_3 and minimum signatures B_1, B_2, B_3 such that for distinct $i, j \in [3]$,*

(C1) $|C_i \cap B_i| \geq 3$ *and* $C_i \cap B_j = \{e\}$,

(C2) $C_i \cap C_j = \{e\} = B_i \cap B_j$,

(C3) the only odd cycles contained in $C_i \cup C_j$ *are* C_i, C_j,

(C4) the only signatures contained in $B_i \cup B_j$ *are* B_i, B_j.

Let x, y be the ends of e. For each $i \in [3]$, let $P_i := C_i - \{e\}$. Notice that P_1, P_2, P_3 are xy-paths that are (edge-)disjoint by (C2).

Claim 2. *For distinct $i, j \in [3]$, P_i and P_j are internally vertex-disjoint xy-paths.*

Proof of Claim. Suppose for a contradiction that P_1, P_2 have a vertex v other than x, y in common. Let $C := P_1[x, v] \cup P_2[v, y] \cup \{e\}$. Observe that C is a cycle, and because for the signature B_3 we have $B_3 \cap C = \{e\}$ by (C1), it follows that C is an odd cycle. However, C is an odd cycle contained in $C_1 \cup C_2$ that is different from C_1, C_2, a contradiction to (C3). Thus, P_1, P_2 are internally vertex-disjoint, and similarly, for distinct $i, j \in [3]$, P_i and P_j are internally vertex-disjoint. ♢

For distinct $i, j \in [3]$, pick $U_{ij} \subseteq V - \{x\}$ such that $B_i \triangle B_j = \delta(U_{ij})$ – as $e \notin B_i \triangle B_j$, it follows that $U_{ij} \subseteq V - \{x, y\}$.

Claim 3. *There are disjoint vertex subsets $U_1, U_2, U_3 \subseteq V$ such that for every permutation i, j, k of $1, 2, 3$,*

(C5) $U_{ij} = U_i \cup U_j$, and

(C6) each edge with an end in U_i and the other in U_j belongs to B_k, each edge with an end in U_k and the other in $V - (U_1 \cup U_2 \cup U_3)$ also belongs to B_k, and $B_k - \{e\}$ has no other edges.

Proof of Claim. Observe that

$$\emptyset = (B_1 \triangle B_2) \triangle (B_2 \triangle B_3) \triangle (B_3 \triangle B_1) = \delta(U_{12}) \triangle \delta(U_{23}) \triangle \delta(U_{31}) = \delta(U_{12} \triangle U_{23} \triangle U_{31}).$$

As G is connected, and $x, y \notin U_{12} \triangle U_{23} \triangle U_{31}$, it follows that $U_{12} \triangle U_{23} \triangle U_{31} = \emptyset$. This implies that there are disjoint vertex subsets $U_1, U_2, U_3 \subseteq V$ such that $U_{ij} = U_i \cup U_j$ for distinct $i, j \in [3]$. This proves **(C5)**. **(C6)** follows from the definition of U_1, U_2, U_3 and the fact (C2) that $B_1 \cap B_2 = B_2 \cap B_3 = B_3 \cap B_1 = \{e\}$. ♢

Claim 4. *For every permutation i, j, k of $1, 2, 3$, we have*

(C7) $V(P_i) \cap (U_j \cup U_k) = \emptyset$ and $V(P_i) \cap U_i \neq \emptyset$, and

(C8) $G[U_i \cup U_j]$ is connected.

Proof of Claim. **(C7)** As $P_i \cap B_j = P_i \cap B_k = \emptyset$, and P_i is an xy-path, it follows from (C6) that $V(P_i) \cap (U_j \cup U_k) = \emptyset$. Moreover, by (C1), $P_i \cap B_i \neq \emptyset$, so $V(P_i) \cap U_i \neq \emptyset$. **(C8)** Suppose otherwise. Then there is a non-empty and proper subset U of $U_i \cup U_j$ such that $\delta(U) \subseteq \delta(U_i \cup U_j) = \delta(U_{ij}) = B_i \triangle B_j$. Moreover, as G is connected, it follows that $\delta(U)$ is a non-empty and proper subset of $B_i \triangle B_j$. Then $B_i \triangle \delta(U)$ is a signature contained in $B_i \cup B_j$, so by (C4), $B_i \triangle \delta(U)$ is either B_i or B_j, implying in turn that $\delta(U)$ is either $\emptyset$ or $B_i \triangle B_j$, a contradiction. ♢

Let $B := B_1 \triangle B_2 \triangle B_3 = B_1 \cup B_2 \cup B_3$. Notice that B is also a signature as $B = B_1 \triangle \delta(U_2 \cup U_3)$, so (G, B) is a resigning of (G, Σ). Let H be the graph obtained from G after contracting all the edges in each $G[U_i]$ and each $C_i - B_i$, and deleting all the remaining edges outside $B_1 \cup B_2 \cup B_3$. Observe that $E(H) = B$, and so $(H, E(H))$ is a minor of (G, Σ). For each $i \in [3]$, let P_i' be an xy-path in $P_i \cap B_i$ and let U_i' be the vertices of H corresponding to the vertices U_i of G. Let $U_0' := V(H) - (U_1' \cup U_2' \cup U_3')$. Notice that P_1', P_2', P_3' are internally vertex-disjoint xy-paths of H, that U_0', U_1', U_2', U_3' form a partition of $V(H)$ into stable sets of $H \setminus e$ by (C6), that for each $i \in [3]$ we have $V(P_i') \subseteq U_0' \cup U_i'$ and $V(P_i') \cap U_i' \neq \emptyset$ by (C7), and for distinct $i, j \in [3]$, $H[U_i' \cup U_j']$ is connected by (C8). In particular, for distinct $i, j \in [3]$, $H[U_i' \cup U_j']$ contains a path with one end in $V(P_i')$ and the other in $V(P_j')$. As a result, $(H, E(H))$ is a pseudo-odd-K_5, so by Theorem 10.9, it has an odd-K_5 minor, implying in turn that (G, Σ) has an odd-K_5 minor, as required. $\square$

As a consequence, we get the following characterization of weakly bipartite graphs:

Corollary 10.11. *Let $G = (V, E)$ be a graph. Then the following statements are equivalent:*

(i) G is not weakly bipartite,

(ii) there exist disjoint $I, J \subseteq E$ such that J forms a cut of $G \setminus I$, and $G \setminus I/J$ is a K_5.

Proof. **(ii) $\Rightarrow$ (i)**: Since J forms a cut of $G \setminus I$, it follows that

$$(K_5, E(K_5)) = (G \setminus I/J, E(G \setminus I/J)) = (G \setminus I, E(G \setminus I))/J = (G, E(G)) \setminus I/J,$$

so $(G, E(G))$ has an odd-K_5 minor, implying by Remark 10.7 that $(G, E(G))$, and so G, is not weakly bipartite. **(i) $\Rightarrow$ (ii)**: It follows that $(G, E(G))$ is not weakly bipartite, so by Theorem 10.10, there are disjoint $I, J \subseteq E$ such that $(K_5, E(K_5)) = (G, E(G)) \setminus I/J$. Let $H := G \setminus I$. Then $(K_5, E(K_5)) = (H, E(H))/J$, so $E(K_5) = E(H) - J$ is a signature of $(H, E(H))$ disjoint from J. As a result, $J = (E(H) - J) \triangle E(H)$ is a cut of H, as required. $\square$

11 Cube-ideal sets

Take an integer $n \geq 1$. We will be working over the hypercube $\{0,1\}^n$. Inequalities of the form

$$1 \geq x_i \geq 0 \qquad i \in [n]$$

are called *hypercube inequalities*. Inequalities of the form

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \qquad \text{for disjoint } I, J \subseteq [n]$$

are called *generalized set covering inequalities*. Notice that generalized set covering inequalities are precisely those inequalities that cut off a sub-hypercube of $\{0,1\}^n$. Take a subset $S \subseteq \{0,1\}^n$. We say that S is *cube-ideal* if its convex hull $\mathrm{conv}(S)$ can be described by hypercube and generalized set covering inequalities. When is a set cube-ideal? This is the theme of this section.

Example. $\{111, 100, 010, 001\} \subseteq \{0,1\}^3$ *is cube-ideal as its convex hull is equal to*

$$\left\{ x \in [0,1]^3 : \begin{array}{ll} x_1 + x_2 + x_3 & \geq 1 \\ x_1 + (1-x_2) + (1-x_3) & \geq 1 \\ (1-x_1) + x_2 + (1-x_3) & \geq 1 \\ (1-x_1) + (1-x_2) + x_3 & \geq 1 \end{array} \right\}.$$

Given two vectors $a, b \in \{0,1\}^n$, let $a \triangle b := a + b \pmod 2$. Given a coordinate $i \in [n]$, to *twist coordinate i of S* is to replace S by

$$S \triangle e_i := \{x \triangle e_i : x \in S\}.$$

So to twist coordinate i is to make the change of variables $x_i \mapsto 1 - x_i$. Since hypercube and generalized set covering inequalities are closed under this change of variables, it follows that,

Remark 11.1. *Take an integer $n \geq 1$ and a subset $S \subseteq \{0,1\}^n$. If S is cube-ideal, then so is any set obtained after twisting some coordinates.*

The *cuboid of S*, denoted $\mathrm{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members have incidence vectors

$$(x_1, 1-x_1, x_2, 1-x_2, \ldots, x_n, 1-x_n) \qquad x \in S.$$

Notice that $\{2i-1, 2i\}, i \in [n]$ are covers of $\mathrm{cuboid}(S)$, and that every member of $\mathrm{cuboid}(S)$ has cardinality n.

Example. *The cuboid of $\{111, 100, 010, 001\} \subseteq \{0,1\}^3$ has incidence matrix*

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix},$$

which is just the incidence matrix of Q_6. Thus, Q_6 is a cuboid.

We saw that $\{111, 100, 010, 001\}$ is cube-ideal, and that its cuboid is Q_6, which we know is an ideal clutter. In fact, we will show next time that in general, a set is cube-ideal if and only if its cuboid is ideal.