# CO 750 Packing and Covering: Lecture 2

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## 2 A review of integral polyhedra and totally dual integral linear systems

Take integers $m, n \geq 1$, a rational $m \times n$ matrix $M$, and a rational $m$-dimensional (column) vector $b$. The set

$$P := \left\{x \in \mathbb{R}^n : Mx \geq b, x \geq \mathbf{0}\right\}$$

is called a *polyhedron*. Hereinafter, $\mathbf{0}$ is the all-zeros vector of appropriate dimension. If $P$ is a bounded set, then it is called a *polytope*. We will always be working with non-empty and full-dimensional polyhedra. A *vertex*, or an *extreme point*, of $P$ is a point $x^\star \in P$ satisfying any of the following equivalent conditions:

- if for $x_1, x_2 \in P$ we have $x^\star = \frac{1}{2}x_1 + \frac{1}{2}x_2$, then $x_1 = x_2 = x^\star$,
- there is a row subsystem $M'x \geq b'$ of $\begin{pmatrix} M \\ I \end{pmatrix} x \geq \begin{pmatrix} b \\ \mathbf{0} \end{pmatrix}$, where $\mathrm{rank}(M') = n$ and $M'x^\star = b'$,
- there exists an integral cost vector $w \in \mathbb{Z}^n$ such that $x^\star$ is the unique optimal solution to the linear program

$$\min \left\{w^\top x : x \in P\right\}.$$

We say that $P$ is *integral* if all its vertices are integral.

For a variable cost vector $w \in \mathbb{Z}^n$, consider the primal linear program

$$\begin{array}{lll} & \min & w^\top x \\ (P) & \text{s.t.} & Mx \geq b \\ & & x \geq \mathbf{0} \end{array}$$

and the dual linear program

$$\begin{array}{lll} & \max & b^\top y \\ (D) & \text{s.t.} & M^\top y \leq w \\ & & y \geq \mathbf{0}. \end{array}$$

By LP Strong Duality, the optimal values of these two programs are equal, whenever the primal (P) is feasible and has a finite optimum. We say that the linear system $Mx \geq b, x \geq \mathbf{0}$ is *totally dual integral (TDI)* if, for all $w \in \mathbb{Z}^n$ for which the primal (P) is feasible and has a finite optimum, the primal (P) and the dual (D) have integral optimal solutions. (Warning: this definition is not standard!) By definition, if $Mx \geq b, x \geq \mathbf{0}$ is TDI, then the polyhedron $\{x \geq \mathbf{0} : Mx \geq b\}$ is integral.

**Theorem 2.1** (Hoffman 1974, Edmonds and Giles 1977)**.** *The following statements are equivalent:*

- $Mx \geq b, x \geq \mathbf{0}$ *is totally dual integral,*
- *for all* $w \in \mathbb{Z}^n$ *for which the primal (P) is feasible and has a finite optimum, the dual (D) has an integral optimal solution.*

# 3 Packing and covering models

There are only two polyhedra that we are interested in. Let $A, B$ be $0-1$ matrices, where $B$ has no column of all zeros. We will call

$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$

the *set covering polyhedron*, and

$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$

the *set packing polytope*. Here, $\mathbf{1}$ is the all-ones vectors of appropriate dimension. When are these polyhedra integral? When are the associated linear systems TDI? These questions will form the underlying theme of the entire course. The short answers are, the questions have been answered for the set packing case, and they are widely open for the set covering case. But first, why are we even interested?

## 3.1 The set covering polyhedron

Let $A$ be a $0-1$ matrix. Consider the *set covering program*

$$(P) \qquad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & Ax \geq \mathbf{1} \\ & x \geq \mathbf{0} \end{array}$$

and its dual

$$(D) \qquad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \leq w \\ & y \geq \mathbf{0} \end{array}$$

for an integral cost vector $w$.[1] Notice that if $w$ has a negative entry, then (P) does not have a finite optimum. We may therefore focus on non-negative cost vectors $w$.

**Packing $st$-paths**. Let $G = (V, E)$ be a graph and take distinct vertices $s, t$. Let $A$ be the $0-1$ matrix whose columns are labeled by $E$ and whose rows are the incidence vectors of $st$-paths. Let $w \in \mathbb{Z}^E_+$. Then the set covering program (P) can be rewritten as

$$\begin{array}{ll} \min & \sum (w_e x_e : e \in E) \\ \text{s.t.} & \sum (x_e : e \in P) \geq 1 \quad \forall\ st\text{-paths } P \\ & x_e \geq 0 \quad \forall e \in E. \end{array}$$

[1] Believe it or not, Fulkerson (1970) called this dual LP the "packing program" for reasons that will become clear soon. Why are we then calling (P) the set covering program? That will become clear in the next section.

Note that every $st$-cut gives a feasible solution to (P). In particular, the minimum *weight* of an $st$-cut is an upper-bound on the optimal value of (P). Let $G_w$ be the graph obtained from $G$ after replacing each edge $e$ by $w_e$ parallel edges. Then the minimum weight of an $st$-cut in $G$ is simply the minimum cardinality of an $st$-cut in $G_w$. Consider now the dual program (D), which may be rewritten as

$$\begin{array}{ll} \max & \sum (y_P : P \text{ is an } st\text{-path}) \\ \text{s.t.} & \sum (y_P : P \text{ is an } st\text{-path such that } e \in P) \leq w_e \quad \forall e \in E \\ & y_P \geq 0 \quad \forall\, st\text{-paths } P. \end{array}$$

Then a packing of $st$-paths in $G_w$ gives a feasible solution to (D). We will think of a packing of $st$-paths in $G_w$ as a *weighted* packing of $st$-paths in $G$ (where each edge $e$ appears in at most $w_e$ many $st$-paths, and where an $st$-path may be packed more than once). Hence, the maximum value of a weighted packing of $st$-paths in $G$ is a lower-bound on the optimal value of (D). It therefore follows from Theorem **??** that,

**Corollary 3.1.** *Let $G$ be a graph and take distinct vertices $s, t$. Then the set covering system corresponding to the $st$-paths of $G$ is totally dual integral. In particular, the set covering polyhedron*

$$\left\{x \in \mathbb{R}^E_+ : \sum (x_e : e \in P) \geq 1 \quad \forall\, st\text{-paths } P\right\}$$

*is integral.*

## 3.2 The set packing polytope

Let $B$ be a $0-1$ matrix without a column of all zeros. Consider the *set packing program*

$$(P) \qquad \begin{array}{ll} \max & w^\top x \\ \text{s.t.} & Bx \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array}$$

and its dual

$$(D) \qquad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & B^\top y \geq w \\ & y \geq \mathbf{0} \end{array}$$

for an integral cost vector $w$.[2] Notice that if $w$ has a negative entry, then the corresponding variable in an optimal solution will always be set to $0$. We may therefore focus on non-negative cost vectors $w$.

**Covering with chains**. Let $(E, \leq)$ be a partially ordered set. Let $B$ be the $0-1$ matrix whose columns are labeled by $E$ and whose rows are the incidence vectors of chains. Then the set packing program (P) can be rewritten as

$$\begin{array}{ll} \max & \sum (w_e x_e : e \in E) \\ \text{s.t.} & \sum (x_e : e \in C) \leq 1 \quad \forall \text{ chains } C \\ & x_e \geq 0 \quad \forall e \in E. \end{array}$$

[2] Fulkerson (1970) called this dual LP the "covering program".

Observe that an antichain gives a feasible solution to (P). In particular, the maximum *weight* of an antichain is a lower-bound on the optimal value of (P). Let $(E_w, \leq)$ be the partially ordered set obtained from $(E, \leq)$ after replacing each element $e$ by $w_e$ pairwise incomparable copies. Then the maximum weight of an antichain of $(E, \leq)$ is simply the maximum cardinality of an antichain of $(E_w, \leq)$. Consider now the dual program (D), rewritten as

$$\begin{array}{ll} \min & \sum (y_C : C \text{ is a chain}) \\ \text{s.t.} & \sum (y_C : C \text{ is a chain such that } e \in C) \geq w_e \quad \forall e \in E \\ & y_C \geq 0 \quad \forall \text{ chains } C. \end{array}$$

Then a covering of $E_w$ with chains gives a feasible solution to (D). We will think of a covering of $E_w$ with chains as a *weighted* covering of $E$ with chains (where each element $e$ is covered at least $w_e$ times, and chains can be used in a covering more than once). Thus, the minimum value of a weighted covering of $E$ with chains is an upper-bound on the optimal value of (D). It therefore follows from Theorem **??** that,

**Corollary 3.2.** *Let $(E, \leq)$ be a partially ordered set. Then the set packing system corresponding to the chains of $(E, \leq)$ is totally dual integral. In particular, the set packing polytope*

$$\left\{ x \in \mathbb{R}_+^E : \sum (x_e : e \in C) \leq 1 \quad \forall \textit{ chains } C \right\}$$

*is integral.*