CO 750 Packing and Covering: Lecture 15

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8.2 T-joins and T-cuts, continued

Today we prove the following theorem. The proof we present is due to Sebő (1987).

Theorem 8.10 (Seymour 1981)**.** *Take a bipartite graph $G = (V, E)$, and a non-empty subset $T \subseteq V$ of even cardinality. Then the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts. That is, the clutter of minimal T-cuts of a bipartite graph packs.*

Proof. We proceed by induction on the number of vertices of G. The base case $|V| = 2$ holds trivially. For the induction step, assume that $|V| \geq 3$. Denote by τ the minimum cardinality of a T-join. We will construct τ disjoint T-cuts. If $\tau = 1$, then we are done. We may therefore assume that $\tau \geq 2$. Among all minimum T-joins, pick the one J whose longest path is the longest compared to the other ones. Define weights $w \in \{-1, 1\}^E$ as follows: for each $e \in J$ set $w_e := -1$, and for each $e \in E - J$ set $w_e := 1$. By Remark 8.8, G has no negative cycle, and as G is bipartite, every cycle has even weight.

Let Q be the longest path contained in J and let u, v be its ends. As Q is the longest path in J, and as G has no negative cycle, it follows that u, v each have degree 1 in J. In particular, $u, v \in \mathrm{odd}(J) = T$. Let $e^\star$ be the edge of Q incident with u. Then $J \cap \delta(u) = \{e^\star\}$.

Claim 1. *If C is a circuit such that $C \cap \delta(u) \neq \emptyset$ and $e^\star \notin C$, then $w(C) \geq 2$.*

Proof of Claim. Suppose otherwise. Since $w(C) \geq 0$ and $w(C)$ is even, it follows that $w(C) = 0$. So $J \triangle C$ is another minimum T-join, and as Q cannot be extended to a longer path in $J \triangle C$, Q and C must share a vertex other than u. Among all the vertices in $V(Q) - \{u\}$ that also belong to $V(C)$, pick the one w that is closest to u on Q. Let Q' be the uw-path in Q; as $e^\star \notin C$, it follows that $Q' \neq \emptyset$ and $Q' \cap C = \emptyset$. Let P_1, P_2 be the two uw-paths partitioning C. Since $w(P_1) + w(P_2) = w(C) = 0$ and $w(Q') < 0$, it follows that one of $P_1 \cup Q', P_2 \cup Q'$ is a negative circuit, a contradiction. $\Diamond$

Claim 2. *u cannot be adjacent to all the other vertices in T.*

Proof of Claim. Suppose otherwise. In particular, u and v are adjacent, and as G has no negative cycle, Q has length 1. Since Q is the longest path in J, it follows that J is a matching, and as $\tau \geq 2$, the matching has an edge other than the edge of Q. Since u is adjacent to the other matched vertices, G has a triangle, a contradiction as G is bipartite. $\Diamond$

Let $(G', T') := (G, T)/\delta(u)$. Notice that G' is still a bipartite graph, and by Claim 2, $T' \neq \emptyset$. Let $J' := J - \delta(u)$. Then J' is a T'-join of G' of length $\tau - 1$. In fact,

Claim 3. *J' is a minimum T'-join of G'.*

Proof of Claim. Define weights $w' \in \{-1, 1\}^{E(G')}$ on the edges of G' as follows: for each $e \in J'$ set $w'(e) := -1$, and for each $e \in E(G') - J'$ set $w'(e) := 1$. Notice that w' is simply the restriction of w to $E - \delta(u) = E(G')$. To prove that J' is a minimum T'-join of G', it suffices by Remark 8.8 to show that G' does not have a negative circuit. To this end, let C' be a circuit of G', and let C be a circuit of G such that $C' \subseteq C \subseteq C' \cup \delta(u)$. If $C = C'$ or $e^\star \in C$, then $w'(C') = w(C) \geq 0$. Otherwise, $C \cap \delta(u) \neq \emptyset$ and $e^\star \notin C$. It therefore follows from Claim 1 that

$$w'(C') = w(C) - 2 \geq 0,$$

as required. $\Diamond$

Thus, by the induction hypothesis, G' has $\tau - 1$ disjoint T-cuts; these are also disjoint T-cuts of G, and together with $\delta(u)$, they give τ disjoint T-cuts in G, thereby completing the induction step. This finishes the proof. $\square$

This result is actually sufficient to guarantee certificates of optimality for minimum T-joins in general graphs:

Theorem 8.11 (Edmonds and Johnson 1970, 1973)**.** *Take a graph $G = (V, E)$ and a non-empty subset $T \subseteq V$ of even cardinality. Denote by $\mathcal{C}$ be the clutter of minimal T-cuts over ground set E. Then the following statements hold:*

(1) For weights $w \in \mathbb{Z}_+^E$ where every cycle has total even weight, the minimum weight of a T-join is equal to the maximum size of a weighted packing of T-cuts:

$$\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w).$$

(2) (Lovász 1975) For arbitrary weights $w \in \mathbb{Z}_+^E$, the minimum weight of a T-join is equal to the maximum value of a half-integral weighted packing of T-cuts:

$$\tau(\mathcal{C}, w) = \max_{2y \in \mathbb{Z}_+^{\mathcal{C}}} \left\{ \mathbf{1}^\top y : \sum (y_C : e \in C \in \mathcal{C}) \leq w_e \ \ \forall e \in E \right\}.$$

(3) The clutter $\mathcal{C}$ of minimal T-cuts is ideal, that is, the polyhedron

$$\left\{ x \geq \mathbf{0} : \sum (x_e : e \in B) \geq 1 \ \ \forall \, T\text{-cuts } B \right\}$$

is integral, and its vertices are the incidence vectors of the minimal T-joins.

Proof. **(1)** If there is a T-join of weight 0, then there is nothing to show. We may therefore assume that the minimum weight of a T-join is non-zero. Let (G', T') be the pair obtained from (G, T) after contracting all edges of weight 0, and for each edge e with $w_e \geq 1$, replacing e by w_e edges in series (the intermediate vertices will not be included in T'). Notice that every cycle C in G corresponds to a cycle in G' of length $w(C)$, and conversely, every cycle C' in G' corresponds to a cycle in G of weight $|C'|$. Thus, since every cycle of G has even weight, it follows that G' is a bipartite graph. Moreover, it is clear that every T-join J in G corresponds to a T'-join in G' of length $w(J)$, and conversely, every T'-join J' in G' corresponds to a T-join in G of weight $|J'|$. In particular, $T' \neq \emptyset$. It therefore follows from Theorem 8.10 that the minimum cardinality of a T'-join in G' is equal to the maximum number of disjoint T'-cuts of G'. As every packing of T'-cuts in G' corresponds to a weighted packing of T-cuts in G, it follows that $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$, as required. **(2)** Take arbitrary weights $w \in \mathbb{Z}_+^E$. It follows from (1) that

$$2\tau(\mathcal{C}, w) = \tau(\mathcal{C}, 2w) = \nu(\mathcal{C}, 2w) = \max_{y \in \mathbb{Z}_+^{\mathcal{C}}} \left\{ \mathbf{1}^\top y : \sum (y_C : e \in C \in \mathcal{C}) \leq 2w_e \;\; \forall e \in E \right\},$$

thereby proving (2). **(3)** follows immediately from (2). $\square$

After applying Theorem 7.8 to part (3), we get the following:

Corollary 8.12. *Take a graph $G = (V, E)$ and a non-empty subset $T \subseteq V$ of even cardinality. Then the clutter of minimal T-joins is ideal. That is, for all weights $w \in \mathbb{Z}_+^E$, the minimum weight of a T-cut is equal to the maximum value of a fractional weighted packing of T-joins.*

Cornuéjols (2001) conjectures that in the above corollary, the minimum weight of T-cut should be equal to the maximum value of a quarter-integral weighted packing of T-joins. In contrast to T-cuts, packing T-joins is a difficult problem. To illustrate this, we need a definition. A 3-*graph* is a connected bridgeless graph $G = (V, E)$ where every vertex has degree 3.

Proposition 8.13. *Let $G = (V, E)$ be a plane 3-graph. Then the following statements are equivalent:*

(i) G has three disjoint perfect matchings, so the clutter of minimal V-joins packs,

(ii) G has two disjoint V-joins,

(iii) G has a 4-face-coloring.

Proof. **(i)** $\Rightarrow$ **(ii)** holds trivially. **(ii)** $\Rightarrow$ **(iii)**: Suppose that G has disjoint minimal V-joins J_1, J_2. Let $G^\star = (V^\star, E)$ be the plane dual of G, and notice that every face of $G^\star$ is a triangle. Notice that the V-cuts of G are in correspondence with the cycles of $G^\star$ bounding an odd number of triangles, implying in turn that the V-cuts of G are in correspondence with the odd cycles of $G^\star$. Since each J_i is a minimal cover of the V-cuts of G, each J_i is also a minimal cover of the odd cycles of $G^\star$, implying in turn that there is a non-empty cut $\delta(U_i), U_i \subseteq V^\star$ of $G^\star$ such that $\delta(U_i) = E - J_i$. Since $J_1 \cap J_2 = \emptyset$, it follows that $U_1 \cap U_2, U_1 \cap \overline{U_2}, \overline{U_1} \cap U_2, \overline{U_1} \cap \overline{U_2}$ are stable sets of $G^\star$, thereby yielding a 4-vertex-coloring of $G^\star$, and hence a 4-face-coloring of G. **(iii)** $\Rightarrow$ **(i)**: Let

$h \in \{(0,0),(0,1),(1,0),(1,1)\}^{\{\text{faces}\}}$ be a 4-face-coloring of G. For each edge e, whose neighboring faces are F_1 and F_2, let

$$g(e) := h(F_1) + h(F_2) \pmod 2.$$

Since F_1, F_2 are adjacent faces, and therefore have different colors, it follows that $g(e) \in \{(0,1),(1,0),(1,1)\}$. Let

$$\begin{aligned} J_1 &:= \{e \in E : g(e) = (0,1)\} \\ J_2 &:= \{e \in E : g(e) = (1,0)\} \\ J_3 &:= \{e \in E : g(e) = (1,1)\}. \end{aligned}$$

We claim that each J_i is a perfect matching. To see this, take an arbitrary vertex v, whose neighboring faces are F_1, F_2, F_3. Then the three edges incident with v have g-values $h(F_1)+h(F_2), h(F_2)+h(F_3), h(F_3)+h(F_1) \pmod 2$. As $h(F_1), h(F_2), h(F_3)$ are pairwise distinct, we get that the g-values of the three edges incident with v are different, so v is indicent with exactly one edge from each J_i. As this is true for each vertex, it follows that each J_i is a perfect matching, as required. $\square$

It is widely known that 4-face-coloring plane 3-graphs is just as general as 4-face-coloring arbitrary plane graphs. Thus, the implication (ii) $\Rightarrow$ (iii) implies that finding just two disjoint T-joins in a graph can be a difficult problem. Appel and Haken (1977), and again Robertson, Sanders, Seymour and Thomas (1996), proved that plane graphs are 4-face-colorable. As a consequence, the implication (iii) $\Rightarrow$ (i) implies that,

Theorem 8.14. *The clutter of minimal T-joins of a planar 3-graph packs.*

This result does not extend to non-planar 3-graphs. For instance, the Petersen graph is a 3-graph whose clutter of minimal T-joins does not pack, as it is not 3-edge-colorable.