CO 750 Packing and Covering: Lecture 10

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6.2 The pluperfect graph theorem

Recall that for a non-negative matrix A without a column of all zeros A, the antiblocker of

$$P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$$

is the set

$$a(P) := \{y \geq \mathbf{0} : x^\top y \leq 1 \ \ \forall x \in P\}.$$

Last time, we showed that

Proposition 6.3. *Let A be a non-negative matrix without a column of all zeros. Let B be the matrix whose rows are the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$. Then B is non-negative, has no column of all zeros, and*

$$a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$$
$$a(a(P)) = P.$$

Next we study the extreme points of the antiblocker. Given vectors x, y of the same dimension, if x is obtained from y after setting some of the coordinates to 0, then we say that x is a *projection* of y.

Proposition 6.4. *Let A be a non-negative matrix and let $P := \{x \in \mathbb{R}^n_+ : Ax \leq \mathbf{1}\}$. Then the following statements hold:*

(1) Let $\bar{x}$ be an extreme point of P for which

$$\bar{x} \leq \sum_{i=1}^{k} \lambda_i x^i$$

for some points $x^1, \ldots, x^k \in P$ and scalars $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$. Then $\bar{x}$ is a projection of each x^i.

(2) Suppose that A has no column of all zeros. Then every extreme point of $a(P)$ is a (possibly trivial) projection of a row of A.

Proof. **(1)** If $\bar{x} = \mathbf{0}$, then we are done. Otherwise, after possibly rearranging the coordinates, we have $\bar{x} = (\bar{z}, \mathbf{0})$ for some $\ell \geq 1$ and $\bar{z} \in \mathbb{R}^\ell$ such that $\bar{z} > \mathbf{0}$. For each $i \in [k]$, denote by z^i the vector consisting of the first ℓ coordinates of x^i. Then

$$\bar{z} \leq \sum_{i=1}^{k} \lambda_i z^i =: z.$$

Notice that z consists of the first ℓ coordinates of $\sum_{i=1}^{k} \lambda_i x^i$. As $\bar{x}$ is an extreme point of P, there is an $\ell \times \ell$ non-singular submatrix E of A such that $E\bar{z} = \mathbf{1}$. On the one hand, as E is non-negative and $z \geq \bar{z}$, it follows that $Ez \geq E\bar{z} = \mathbf{1}$. On the other hand, as $Ax \leq \mathbf{1}$, it follows that $Ez \leq \mathbf{1}$. Thus, $Ez = E\bar{z} = \mathbf{1}$, implying in turn that $z = \bar{z}$. As a result,

$$\bar{x} = (\bar{z}, \mathbf{0}) = (z, \mathbf{0}) = \sum_{i=1}^{k} \lambda_i (z^i, \mathbf{0}).$$

Since $\bar{x}$ is an extreme point, and each $(z^i, \mathbf{0})$ belongs to P, it follows that $\bar{x} = (z^1, \mathbf{0}) = \cdots = (z^k, \mathbf{0})$, as required.

(2) Denote by B the matrix whose rows are the extreme points of the polytope P. Then by Proposition 6.3, B is a non-negative matrix without a column of all zeros, and $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Denote by A' the matrix whose rows are the extreme points of the polytope $a(P)$. Then by Proposition 6.3,

$$\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\} = a(a(P)) = \{x \geq \mathbf{0} : A'x \leq \mathbf{1}\}.$$

Take an extreme point a' of $a(P)$, which is also a row of A'. Since $a'^\top x \leq \mathbf{1}$ is valid for $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$, it follows that a' is bounded above by a convex combination of the rows of A. Applying (1) to $a(P)$, we see that a' must be a projection of a row of A, as required. $\square$

We are now ready for the pluperfect graph theorem:

Theorem 6.5 (Fulkerson 1972)**.** *Let A be a non-negative matrix without a column of all zeros, and let B be the matrix whose rows are the extreme points of $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$. If A is perfect, then so is B.*

Proof. Suppose that A is perfect, that is, A is a $0-1$ matrix whose associated set packing polytope $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral. So B is a $0-1$ matrix. By Proposition 6.3, B has no column of all zeros and $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Therefore, by Proposition 6.4 (2), every extreme point of $\{y \geq \mathbf{0} : By \leq \mathbf{1}\}$ is a projection of a row of A. In particular, $\{y \geq \mathbf{0} : By \leq \mathbf{1}\}$ is integral, that is, B is perfect. $\square$

6.3 Clutters and antiblockers

Let V be a finite set of *elements*, and let $\mathcal{A}$ be a family of subsets of V, called *members*. We say that $\mathcal{A}$ is a *clutter* over *ground set* V if no member is contained in another one. The *incidence matrix* of $\mathcal{A}$, denoted $M(\mathcal{A})$, is the $0-1$ matrix whose columns are labeled by V and whose rows are the incidence vectors of the members.

Remark 6.6. *Let $\mathcal{A}_1, \mathcal{A}_2$ be clutters over the same ground set, where every member of $\mathcal{A}_1$ contains a member of $\mathcal{A}_2$, and every member of $\mathcal{A}_2$ contains a member of $\mathcal{A}_1$. Then $\mathcal{A}_1 = \mathcal{A}_2$.*

Proof. Take $A_1 \in \mathcal{A}_1$. Then A_1 contains a member A of $\mathcal{A}_2$, and A contains a member of $\mathcal{A}_1$. As $\mathcal{A}_1$ is a clutter, it must be that $A_1 \subseteq A \subseteq A_1$, implying in turn that $A = A_1$. Thus, $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Similarly, $\mathcal{A}_2 \subseteq \mathcal{A}_1$, so $\mathcal{A}_1 = \mathcal{A}_2$. $\square$

Let $\mathcal{A}$ be a clutter over ground set V, where every element is contained in a member. Consider the set packing polytope associated with $\mathcal{A}$:

$$\left\{x \in \mathbb{R}^V_+ : \sum (x_v : v \in A) \leq 1 \ \ \forall A \in \mathcal{A}\right\} = \{x \geq \mathbf{0} : M(\mathcal{A})x \leq \mathbf{1}\}.$$

Notice that the $0-1$ points of $P(\mathcal{A})$ correspond to the sets in

$$\{B \subseteq V : |B \cap A| \leq 1 \ \ \forall A \in \mathcal{A}\},$$

and that every $0-1$ point of the polytope is in fact an extreme point. We say that $\mathcal{A}$ is a *perfect clutter* if the associated set packing polytope is integral, that is, when the associated incidence matrix $M(\mathcal{A})$ is perfect. Notice that an arbitrary $0-1$ matrix A is perfect if, and only if, the clutter corresponding to the maximal rows of A is perfect. As a consequence, studying perfect clutters is just as general as studying perfect matrices.

Let $\mathcal{A}$ be a clutter over ground set V. The maximal sets of $\{B \subseteq V : |B \cap A| \leq 1 \ \ \forall A \in \mathcal{A}\}$ form another clutter over the same ground set, called the *antiblocker of* $\mathcal{A}$ and denoted $a(\mathcal{A})$. If every element is used in a member of $\mathcal{A}$, then the members of $a(\mathcal{A})$ are precisely the maximal integral points contained in the set packing polytope. For instance,

$$\begin{aligned}
\text{the antiblocker of } \{\{1,2\},\{2,3\},\{3,1\}\} &= \{\{1\},\{2\},\{3\}\} \\
\text{the antiblocker of } \{\{1\},\{2\},\{3\}\} &= \{\{1,2,3\}\} \\
\text{the antiblocker of } \{\{1,2,3\}\} &= \{\{1\},\{2\},\{3\}\}.
\end{aligned}$$

One natural question to ask is, when do we have $a(a(\mathcal{A})) = \mathcal{A}$? Perhaps surprisingly, the answer is very simple:

Proposition 6.7 (Fulkerson 1971)**.** *Let $\mathcal{A}$ be a clutter over ground set V. Then the following statements are equivalent:*

(i) $a(a(\mathcal{A})) = \mathcal{A}$*,*

(ii) *$\mathcal{A}$ consists of the maximal stable sets of a graph over vertex set V.*

Proof. **(ii)** $\Rightarrow$ **(i)**: Suppose $\mathcal{A}$ consists of the maximal stable sets of $G = (V, E)$. Then a vertex set intersects every stable set at most once if, and only if, it is a clique. This implies that $a(\mathcal{A})$ consists of the maximal cliques of G. Applying the same argument to $\overline{G}$ implies that $a(a(\mathcal{A}))$ consists of the maximal stable sets of G, so $a(a(\mathcal{A})) = \mathcal{A}$. **(i)** $\Rightarrow$ **(ii)**: Suppose $a(a(\mathcal{A})) = \mathcal{A}$. Let G be the graph over vertex set V, where distinct vertices u, v are non-adjacent if there is a member containing both u, v. Clearly, every member of $\mathcal{A}$ is a stable set of G. Conversely, let $S \subseteq V$ be a stable set of G. We claim that

$$(\star) \qquad |S \cap B| \leq 1 \quad \forall B \in a(\mathcal{A}).$$

Suppose otherwise. Then for distinct vertices u, v of G, $\{u, v\} \subseteq S \cap B$. However, as u and v are non-adjacent, $\{u, v\} \subseteq A$ for some member $A \in \mathcal{A}$, but then $\{u, v\} \subseteq A \cap B$, a contradiction as $B \in a(\mathcal{A})$. This proves $(\star)$, implying in turn that S is contained in a member of $a(a(\mathcal{A})) = \mathcal{A}$. Remark 6.6 implies that $\mathcal{A}$ consists of the maximal stable sets of G, as required. $\square$

As a consequence,

Theorem 6.8 (Padberg 1973)**.** *If a clutter is perfect, then its members are the maximal stable sets of a simple graph.*

Proof. Let $\mathcal{A}$ be a perfect clutter over ground set V, and let A be the corresponding incidence matrix. Let B be the matrix whose rows are the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$, and let $Q := \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Then by Proposition 6.3, $a(P) = Q$ and $a(Q) = P$. Moreover, since the clutter $\mathcal{A}$ is perfect, the matrix A is perfect, so by Theorem 6.5, B is a perfect matrix. Let $\mathcal{B}$ be the clutter over ground set V whose members correspond to the maximal rows of B. Notice that $a(\mathcal{A})$ corresponds to the maximal integral extreme points of P, so $a(\mathcal{A}) = \mathcal{B}$. Similarly, $a(\mathcal{B})$ corresponds to the maximal integral extreme points of Q, so $a(\mathcal{B}) = \mathcal{A}$. It therefore follows from Proposition 6.7 that $\mathcal{A}$ consists of the maximal stable sets of a graph, as required. $\square$

In fact, as we will see on Assignment 2, the simple graph above is perfect:

Theorem 6.9 (Chvátal 1975)**.** *Let $G = (V, E)$ be a simple graph. If the clutter of the maximal stable sets of G is perfect, then G is a perfect graph.*

Summarizing the results of this section and the previous one, we get the following characterization of when the set packing polytope is integral:

Corollary 6.10. *The following statements hold:*

(1) Let A be a $0-1$ matrix without a column of all zeros whose set packing polytope $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral. Then the linear system $x \geq \mathbf{0}, Ax \leq \mathbf{1}$ is totally dual integral, the maximal rows of A correspond to the maximal stable sets of a simple graph, and the graph is perfect.

(2) Let G be a simple graph. Then G is perfect if, and only if, it has no odd hole and no odd antihole.