CO 750 Packing and Covering: Lecture 1

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1 What is packing and covering?

1.1 A packing example: Menger's theorem

Let $G = (V, E)$ be a loopless graph, and take distinct vertices $s, t \in V$. An *st-path* is a minimal edge subset connecting s and t. What is the maximum number of (pairwise) disjoint st-paths? In other words, how many st-paths can we *pack*? Denote by ν the maximum number of disjoint st-paths.

An *st-cut* is an edge subset of the form

$$\delta(U) := \{e \in E : |e \cap U| = 1\}$$

where $U \subseteq V$ satisfies $U \cap \{s, t\} = \{s\}$. We will refer to U and $V - U$ as the *shores* of G. Notice that every st-path intersects an st-cut. Thus, ν is at most the cardinality of any st-cut. Let τ be the minimum cardinality of an st-cut. Then

$$\tau \geq \nu.$$

Theorem 1.1 (Menger 1927)**.** *Let $G = (V, E)$ be a loopless graph, and take distinct vertices $s, t \in V$. Then the maximum number of disjoint st-paths is equal to the minimum cardinality of an st-cut, that is, $\tau = \nu$.*

Proof. We prove this by induction on $|V| + |E| \geq 3$. The result is obvious for $|V| + |E| = 3$. For the induction step, assume that $|V| + |E| \geq 4$. Let τ be the minimum cardinality of an st-cut. We may assume that $\tau \geq 1$. We will find τ disjoint st-paths.

Claim 1. *If an edge e does not appear in a minimum st-cut, then G has τ disjoint paths.*

Proof of Claim. Notice that the cardinality of a minimum st-cut in $G \setminus e$ is still τ. As a result, the induction hypothesis implies the existence of τ disjoint st-paths in $G \setminus e$, and therefore in G. $\Diamond$

We may therefore assume that every edge appears in a minimum st-cut. An st-cut $\delta(U)$ is *trivial* if either $|U| = 1$ or $|V - U| = 1$.

Claim 2. *If there is a minimum st-cut which is not trivial, then G has τ disjoint paths.*

Proof of Claim. Let $\delta(U), s \in U \subseteq V - \{t\}$ be a minimum st-cut which is non-trivial. Let G_1 be the graph obtained from G by shrinking U to a single vertex s', and let G_2 be the graph obtained from G after shrinking $V - U$ to a single vertex t'. Since $\delta(U)$ is non-trivial, it follows that $|V(G_i)| + |E(G_i)| < |V| + |E|$, for each $i \in [2]$. We may therefore apply the induction hypothesis to G_1 and G_2. Notice that τ is still the minimum cardinality of an $s't$-cut in G_1 and of an st'-cut in G_2. Thus, by the induction hypothesis, G_1 has τ disjoint $s't$-paths and G_2 has disjoint st'-paths. Gluing these paths along the edges of $\delta(U)$ gives us τ disjoint st-paths in G. $\Diamond$

We may therefore assume that every minimum st-cut is trivial. Since every edge appears in a minimum st-cut, it follows that every edge has either s or t as an end. In this case, G has a special form and it is clear that $\tau = \nu$ for this graph, thereby completing the induction step. $\square$

This result was the first of many packing theorems. Just to mention a few, we will see some of these packing results:

- Given a connected loopless graph G and distinct vertices s, t, the maximum number of disjoint st-cuts is equal to the minimum cardinality of an st-path.
- Ford and Fulkerson 1956: given a directed graph G and distinct vertices s, t, the maximum number of disjoint directed (s, t)-paths is equal to the minimum cardinality of an (s, t)-cut.
- Edmonds 1972: given a directed graph G and a root r, the maximum number of disjoint spanning r-arborescences is equal to the minimum cardinality of an r-cut.
- Edmonds and Johnson 1973: given a graph G and even subset T of vertices, the maximum value of a fractional packing of T-joins is equal to the minimum cardinality of a T-cut.
- Lucchesi and Younger 1976: given a directed graph G, the maximum number of disjoint dicuts is equal to the minimum cardinality of a dijoin.
- Conjecture (Woodall 1978): given a directed graph G, the maximum number of disjoint dijoins is equal to the minimum cardinality of a dicut.
- Guenin 2001: in a signed graph without an odd-K_5 minor, the maximum value of a fractional packing of odd circuits is equal to the minimum cardinality of a signature.

1.2 A covering example: Dilworth's theorem

Take a partially ordered set $(E, \leq)$, that is, the following statements hold for all $a, b, c \in E$:

- $a \leq a$,
- if $a \leq b$ and $b \leq a$, then $a = b$,

- if $a \le b$ and $b \le c$, then $a \le c$.

We say that a, b are *comparable* if $a \ge b$ or $b \ge a$; otherwise they are *incomparable*. A *chain* is a set of pairwise comparable elements. What is the minimum number of (not necessarily disjoint) chains whose union is E? That is, what is the least number of chains needed to *cover* the ground set? Let θ be the minimum size of a covering.

An *antichain* is a set of pairwise incomparable elements. Given an antichain A, every chain picks at most one element from A. Thus, θ is at least the cardinality of an antichain. Let α be the maximum cardinality of an antichain. Then

$$\theta \ge \alpha.$$

Theorem 1.2 (Dilworth 1950)**.** *Let $(E, \le)$ be a partially ordered set. Then the minimum number of chains needed to cover E is equal to the maximum cardinality of an antichain. That is, $\theta = \alpha$.*

Proof. We prove this by induction on $|E|$. The base case $|E| = 1$ is obvious. For the induction step, assume that $|E| \ge 2$. Let α be the maximum cardinality of an antichain. We will find α chains covering E. If $\alpha = |E|$, then $\theta = \alpha = |E|$ and we are done. Otherwise, $\alpha < |E|$, implying in turn that there is a chain $\{a, b\}$ where a is a minimal element and b is a maximal element. Let $E' := E - \{a, b\}$.

Claim. *If the maximum cardinality of an antichain of $(E', \le)$ is $\alpha - 1$, then there are α chains covering E.*

Proof of Claim. By the induction hypothesis, there are $\alpha - 1$ chains of E' covering $E - \{a, b\}$. Together with $\{a, b\}$, we get a covering of E using α chains. $\Diamond$

We may therefore assume that E' has an antichain A such that $|A| = \alpha$. Let

$$E^+ := A \cup \{x \in E - A : x \ge z \text{ for some } z \in A\}$$
$$E^- := A \cup \{y \in E - A : y \le z \text{ for some } z \in A\}.$$

Since A is an antichain, $E^+ \cap E^- = A$, and since it is a maximum antichain, $E^+ \cup E^- = E$. As a is minimal and $a \notin A$, it follows that $a \notin E^+$. As b is maximal and $b \notin A$, we get that $b \notin E^-$. In particular, $|E^+|, |E^-| < |E|$. Thus, by the induction hypothesis, E^+ has α chains covering it, and E^- has α chains covering it. Gluing these chains together, we get α chains covering $E^+ \cup E^- = E$, thereby completing the induction step. $\square$

This result was the first of many covering results. To name a few:

- In a partially ordered set, the minimum number of antichains needed to cover the ground set is equal to the maximum cardinality of a chain.
- Kőnig 1931: In a bipartite graph, the minimum number of colors needed for an edge-coloring is equal to the maximum degree of a vertex.
- Kőnig 1931: In a bipartite graph, the minimum number of vertices needed to cover the edges is equal to the maximum cardinality of a matching.

- Gallai 1962, Suranyi 1968: In a chordal graph, the minimum number of cliques needed to cover the vertices is equal to the maximum cardinality of a stable set.
- Sachs 1970: In a chordal graph, the minimum number of colors needed for a vertex-coloring is equal to the maximum cardinality of a clique.
- Chudnovski, Robertson, Thomas and Seymour 2006: In a graph without an odd hole or an odd hole complement, the minimum number of cliques needed to cover the vertices is equal to the maximum cardinality of a stable set.