CO 750-1 Assignment 2

Due Thursday, June 8, in class

The assignment will be out of 35 points. If you get more than 35 points, the extra points will be carried over to your future assignments with half the weight. You may use the results in a problem for proving another problem, even if you didn't solve the first problem.

1. **(5 points)** Let $G = (V, E)$ be a simple graph. Prove the following statements:
   (a) If G is a triangle-free perfect graph, then it is bipartite.
   (b) If G is a minimally imperfect graph with a vertex of degree 2, then it is an odd hole.
2. **(5 points)** Prove that a double split graph is perfect.
3. **(5 points)** Take perfect graphs G_1, G_2 over disjoint vertex sets. Let C_1, C_2 be cliques of G_1, G_2 such that $|C_1| = |C_2|$. Let G be the graph obtained from G_1, G_2 after identifying the two cliques C_1, C_2. Prove that G is perfect. (**Hint**. The Star Cutset Lemma)
4. **(10 points)** A *hole* is an induced subgraph that is a circuit on at least 4 vertices. Let $G = (V, E)$ be a simple graph without a hole.
   (a) Prove that every circuit on at least 4 vertices has a chord.
   (b) A *clique cutset* is a non-empty subset $X \subseteq V$ such that $G \setminus X$ is not connected and $G[X]$ is a clique. Prove that G is either a clique or has a clique cutset. (**Hint**. Show that every minimal cutset, if any, is a clique.)
   (c) Prove that G is perfect.
5. **(5 points)** Let $G = (V, E)$ be a graph, and let A be the $0 - 1$ matrix whose columns are labeled by V and whose rows are the stable sets of G. Prove that if A is a perfect matrix, then G is a perfect graph.
6. **(5 points)** Use the Star Cutset Lemma to prove Dilworth's theorem.
7. **(5 points)** Prove that a minimally imperfect graph does not have a pair u, v of distinct vertices where u *dominates* v, that is, $N(v) - \{u\} \subseteq N(u) - \{v\}$.

8. **(21 points)** Let $G = (V, E)$ be a simple graph. For a non-empty set $X \subseteq V$ and a vertex $v \in V - X$, we say that v is X-*universal* if v is adjacent to all of X, that v is X-*null* if v is adjacent to none of X, and that v is X-*partial* if it is neither X-universal nor X-null. Recall that a pair (X_1, X_2) of disjoint non-empty vertex subsets is *homogeneous* if $|X_1| + |X_2| \geq 3$, $|V - (X_1 \cup X_2)| \geq 2$, all X_1-partial vertices are in X_2, and all X_2-partial vertices are in X_1.

   In this question, we will prove that a minimally imperfect graph $G = (V, E)$ does not have a homogeneous pair.[1] Suppose for a contradiction that (X_1, X_2) is a homogeneous pair.

   (a) Prove that X_1 contains a vertex that is not X_2-null and a vertex that is not X_2-universal.

   (b) Prove that there is an X_1-universal X_2-null vertex, and that there is an X_1-null X_2-universal vertex.

   Let H be the graph obtained from $G[V - (X_1 \cup X_2)]$ after adding vertices u_1, v_1, u_2, v_2, and edges $\{u_1, v_2\}, \{v_2, v_1\}, \{v_1, u_2\}$ and $\{w, u_i\}, \{w, v_i\}$ for each $i \in [2]$ and X_i-universal vertex $w \in V - (X_1 \cup X_2)$.

   (c) Prove that H is perfect. (**Hint**. Problem 7)

   Consider the following costs defined on the vertices of H: for each $v \in V - (X_1 \cup X_2)$ let $c(v) := 1$, and let

$$c(u_1) = \omega(X_1) \quad c(v_1) = \omega(X_1 \cup X_2) - \omega(X_2)$$
$$c(u_1) = \omega(X_2) \quad c(v_2) = \omega(X_1 \cup X_2) - \omega(X_1).$$

   Let $C_H \subseteq V(H)$ be a clique in H of maximum cost k.

   (d) Transform C_H into a clique of G of cardinality k.

   Let $S_H^1, \ldots, S_H^k$ be stable sets in H where every vertex v is covered by exactly $c(v)$ stable sets. We will use these stable sets to find a k-vertex-coloring of G. Let

$$I_0 := \{i \in [k] : S_H^i \cap \{u_1, v_1\} = \emptyset, S_H^i \cap \{u_2, v_2\} = \emptyset\}$$
$$I_1 := \{i \in [k] : S_H^i \cap \{u_1, v_1\} \neq \emptyset, S_H^i \cap \{u_2, v_2\} = \emptyset\}$$
$$I_2 := \{i \in [k] : S_H^i \cap \{u_1, v_1\} = \emptyset, S_H^i \cap \{u_2, v_2\} \neq \emptyset\}$$
$$I_3 := \{i \in [k] : S_H^i \cap \{u_1, v_1\} \neq \emptyset, S_H^i \cap \{u_2, v_2\} \neq \emptyset\}.$$

   Let F be the graph obtained from $G[X_1 \cup X_2]$ after adding adjacent vertices x_1, x_2, and an edge $\{x_i, y\}$ for each $i \in [2]$ and $y \in X_i$.

[1] Our proof follows Chvátal and Sbihi (1987). I have broken down their proof into 7 reasonable parts, and each is worth 3 points. Please do not look up the proof.

(e) Prove that F is perfect. (**Hint**. The Antitwin Lemma)

Define the following costs on the vertices of F: for each $v \in X_1 \cup X_2$ let $d(v) := 1$, and let $d(x_1) := |I_2|$ and $d(x_2) := |I_1|$.

(f) Find stable sets $S_F^1, \ldots, S_F^{|I_1|+|I_2|+|I_3|}$ in F where every vertex v is covered by exactly $d(v)$ stable sets.

(g) Use $S_H^1, \ldots, S_H^k$ and $S_F^1, \ldots, S_F^{|I_1|+|I_2|+|I_3|}$ to find a k-vertex-coloring of G.

As a result, $k \geq \chi(G) > \omega(G) \geq k$, a contradiction. Hence, a minimally imperfect graph does not have a homogeneous pair.