47853 Packing and Covering: Lecture 5

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January 29, 2019

5 Perfect graphs

Let $G = (V, E)$ be a simple graph. Recall that G is perfect if, for every induced subgraph G' of G, $\chi(G') = \omega(G')$. (Notice that G' may be G.) In words, a simple graph is perfect if in each induced subgraph, the maximum cardinality of a clique is equal to the chromatic number. Last time we proved the following:

Corollary 5.4. *The following graphs are perfect:*

(1) bipartite graphs, and their complements,

(2) line graphs of bipartite graphs, and their complements,

(3) comparability graphs, and their complements.

The obvious question this corollary leads to is, does complementation preserve perfection? Claude Berge asked the same question in 1961 [1]. Today we will see that the answer is surprisingly yes!

5.1 The max-max inequality and the weak perfect graph theorem

The proof we present of the following result is due to Gasparian 1996 [4]:

Theorem 5.5 (Lovász 1972 [5])**.** *Let G be a simple graph. The following statements are equivalent:*

(i) G is perfect,

(ii) $\omega(H) \cdot \alpha(H) \geq |V(H)|$ for every induced subgraph H.

Proof. **(i) ⇒ (ii)**: Let H be an induced subgraph. By definition, $\chi(H) = \omega(H)$, that is, $V(H)$ can be covered by $\omega(H)$ stable sets. Since each stable set has cardinality at most $\alpha(H)$, it follows that

$$|V(H)| \leq \omega(H) \cdot \alpha(H).$$

(ii) $\Rightarrow$ (i): Suppose for a contradiction that G is not perfect. Let H be an induced subgraph of G that is not perfect, but every proper induced subgraph of H is perfect. Let $\omega := \omega(H)$, $\alpha := \alpha(H)$ and $n := |V(H)|$. Note that $n > 1$. Clearly,

$$\omega \geq \omega(H \setminus S) \geq \omega - 1 \qquad \text{for every nonempty stable set } S \subseteq V(H);$$

since $H \setminus S$ is perfect and H is not, it follows that

$$\omega(H \setminus S) = \omega \qquad \text{for every nonempty stable set } S \subseteq V(H).$$

Let S_0 be a maximum stable set of H. Then for every vertex $v \in S_0$, $H \setminus v$ is perfect, so its vertices can be partitioned into $\omega(H \setminus v) = \omega$ nonempty stable sets. As S_0 has α vertices, we get $\alpha\omega$ stable sets $S_1, \ldots, S_{\alpha\omega}$.

Claim. *Every maximum clique of H intersects all but one of $S_0, S_1, \ldots, S_{\alpha\omega}$ exactly once.*

Proof of Claim. Let C be a maximum clique of H. Clearly C intersects each one of $S_0, S_1, \ldots, S_{\alpha\omega}$ at most once. For a vertex $v \in S_0$, if

- $v \in C$: then C intersects all but one stable set in every partition of $V(H \setminus v)$ into ω stable sets,
- $v \notin C$: then C intersects all stable sets in every partition of $V(H \setminus v)$ into ω stable sets.

This observation immediately implies the claim. $\Diamond$

For each $i \in \{0, 1, \ldots, \alpha\omega\}$, let C_i be a maximum clique of $H \setminus S_i$; notice that $|C_i| = \omega$. Let A be the $0-1$ matrix whose columns are labeled by $V(H)$, and whose rows are the incidence vectors of $S_0, S_1, \ldots, S_{\alpha\omega}$. Let B be the $0-1$ matrix whose columns are labeled by $V(H)$, and whose rows are the incidence vectors of $C_0, C_1, \ldots, C_{\alpha\omega}$. It then follows from the claim above that $AB^\top = J - I$, where J is the all-ones matrix and I the identity matrix of appropriate dimensions. Since $J - I$ is a nonsingular $(\alpha\omega + 1) \times (\alpha\omega + 1)$ matrix, it follows that both A and B have full row rank, implying in turn that

$$|V(H)| = n \geq \alpha\omega + 1 = \alpha(H) \cdot \omega(H) + 1 > |V(H)|,$$

a contradiction. $\square$

As a consequence, we get the *weak perfect graph theorem*:

Theorem 5.6 (Lovász 1972 [6])**.** *If a graph is perfect, then so is its complement.*

Proof. Suppose that G is perfect. Then by Theorem 5.5, for every induced subgraph H of G,

$$\omega(H) \cdot \alpha(H) \geq |V(H)|,$$

implying in turn that for every induced subgraph $\overline{H}$ of $\overline{G}$,

$$\alpha(\overline{H}) \cdot \omega(\overline{H}) \geq |V(\overline{H})|,$$

so by Theorem 5.5, $\overline{G}$ is perfect, as required. $\square$

5.2 Odd holes and odd antiholes

We say that a simple graph is *minimally imperfect* if it is not perfect, but every proper induced subgraph is perfect. Equivalently, a simple graph G is minimally imperfect if $\chi(G) > \omega(G)$, but for every proper induced subgraph G', $\chi(G') = \omega(G')$. The latter implies that a minimally imperfect graph is always connected.

Remark 5.7. *A graph is perfect if, and only if, it has no minimally imperfect induced subgraph.*

Let H be an odd circuit with at least 5 vertices. Then $3 = \chi(H) > \omega(H) = 2$, so G is imperfect. Since every proper induced subgraph of H is bipartite, and therefore perfect, it follows that H is minimally imperfect. Notice that Theorem 5.6 equivalently states that,

Corollary 5.8. *The complement of a minimally imperfect graph is also minimally imperfect.*

Thus, the complement of an odd circuit with at least 5 vertices is also minimally imperfect. Let G be a simple graph. We say that G has an *odd hole* if it has as an induced subgraph an odd circuit with at least 5 vertices, and we say that G has an *odd antihole* if $\overline{G}$ has an odd hole. It follows from the preceding remark that,

Remark 5.9. *A perfect graph has no odd hole and no odd antihole.*

In 1961, Claude Berge conjectured that the converse of this statement is also true [1]. In 2006, this conjecture was proved by Chudnovsky, Robertson, Seymour and Thomas, and their theorem is referred to as the *strong perfect graph theorem* [2]. We will see some of the milestones and highlights leading to the proof, as well as a sketch of the proof.

5.3 Star cutsets and antitwins

Let $G = (V, E)$ be a simple graph. A *star cutset* is a nonempty $X \subseteq V$ such that

- $G \setminus X$ has more connected components than G, and
- a vertex of X is adjacent to all the other vertices in X.

Lemma 5.10 (Chvátal 1985 [3])**.** *A minimally imperfect graph does not have a star cutset.*

Proof. Let $G = (V, E)$ be a minimally imperfect graph, and let $\omega := \omega(G)$. Then

$$\omega(G \setminus S) = \omega \quad \text{for every stable set } S \subseteq V.$$

Suppose for a contradiction that G has a star cutset $X \subseteq V$. Then the vertices of $G \setminus X$ can be partitioned into nonempty parts V_1, V_2 such that G has no edge between V_1 and V_2. Since every proper induced subgraph of G is perfect, for each $i \in [2]$, there is a vertex-coloring $f_i : X \cup V_i \to [\omega]$ of the induced subgraph $G[X \cup V_i]$. Since X is a star cutset, it has a vertex v that is adjacent to all other vertices of X. For $i \in [2]$, let $S_i := \{w \in X \cup V_i : f_i(w) = f_i(v)\}$. Clearly, each S_i is stable and $S_i \cap X = \{v\}$. Moreover, since there are no edges between V_1 and V_2, it follows that $S := S_1 \cup S_2$ is also stable. In particular, $\omega(G \setminus S) = \omega$, so $G \setminus S$ has a clique C of cardinality ω. However, either $C \subseteq X \cup V_1$ or $C \subseteq X \cup V_2$, implying in turn that C is an ω-clique of some $G[X \cup V_i] \setminus S_i$, which has an $(\omega - 1)$-vertex-coloring, a contradiction. □

This lemma was a key milestone for what led to the proof of the strong perfect graph theorem. To demonstrate the power of this lemma, let us see some applications of it. Let G_1 be a perfect graph, and take a vertex $v \in V(G_1)$. To *duplicate* v is to introduce a new vertex $\bar{v}$, join it to all the neighbors of v, and then join it to $\bar{v}$. More generally, given another perfect graph G_2 over a disjoint vertex set, to *substitute* G_2 *for* v is to remove v, and join every vertex of G_2 to all the neighbors of v in $G_1 \setminus v$.

Theorem 5.11 (Lovász 1972 [6])**.** *Let G_1, G_2 be perfect graphs over disjoint vertex sets. If G is obtained by substituting G_2 for a vertex v of G_1, then G is perfect. In particular, duplication preserves perfection.*

Proof. Suppose otherwise. Since every induced subgraph of G is either an induced subgraph of G_1, or of G_2, or arises from induced subgraphs of G_1, G_2 by substitution, we may assume that G is minimally imperfect. Clearly, G_2 has at least two vertices, and $G_1 \setminus v$ has at least one vertex. Take an arbitrary vertex u of G_2, and denote by N its neighbors of G in $V(G_1 \setminus v)$. Notice that for each vertex in $V(G_2)$, its neighbors of G in $V(G_1 \setminus v)$ is precisely N. As G is minimally imperfect, $\overline{G}$ is minimally imperfect by Corollary 5.8, so $\overline{G}$ is connected, implying in turn that $V(G_1 \setminus v) - N \neq \emptyset$. Let $X := \{u\} \cup N$. Then X is a star cutset as u is adjacent to all the vertices in N, and in $G \setminus X$, there are no edges between $V(G_2) - \{u\}$ and $V(G_1 \setminus v) - N$. This contradicts the Star Cutset Lemma 5.10. □

Next time we will see another important lemma about minimally imperfect graphs.

References

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