47853 Packing and Covering: Lecture 4

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4 Balanced matrices

Recall the following theorem we proved last time:

Theorem 4.5 (Fulkerson, Hoffman, Oppenheim 1974 [3])**.** *Let* $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ *be a balanced matrix. Then the polyhedron*

$$P = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}, Bx \leq \mathbf{1}, Cx = \mathbf{1}\}$$

is integral. In particular, the set packing polytope and the set covering polyhedron corresponding to a balanced matrix are both integral.

We will need this result today.

4.3 Hall's theorem for balanced hypergraphs

Let $G = (V, E)$ be a hypergraph. A *matching* is a packing of pairwise disjoint edges. A *perfect matching* is a matching that uses every vertex. Recall Hall's condition for the existence of perfect matchings in bipartite graphs:

Theorem 4.6 (Hall 1935 [4])**.** *Let G be a bipartite graph. Then the following statements are equivalent:*

- *G has no perfect matching,*
- *there exist disjoint vertex sets R, B such that $|R| > |B|$ and every edge with an end in R has an end in B.*

We will see a generalization of this to balanced hypergraphs. We will need two lemmas.

Lemma 4.7. *Let A be a balanced matrix. Then the polyhedron*

$$P = \{x, s, t \geq \mathbf{0} : Ax + Is - It = \mathbf{1}\}$$

is integral.

Proof. Denote by m the number of rows of A, and for each $i \in [m]$, denote by a_i the i^{th} row of A. Take an extreme point $(x^\star, s^\star, t^\star)$ of P. Since the corresponding columns of $\begin{pmatrix} A & I & -I \end{pmatrix}$ are linearly dependent, we see that $s_i^\star t_i^\star = 0$ for each $i \in [m]$. As a result, $x^\star$ is also an extreme point of the polyhedron

$$\left\{ x \geq \mathbf{0} : \begin{array}{lll} a_i^\top x & \leq 1 & \forall i \in [m] \text{ s.t. } s_i^\star > 0 \\ a_i^\top x & \geq 1 & \forall i \in [m] \text{ s.t. } t_i^\star > 0 \\ a_i^\top x & = 1 & \text{otherwise.} \end{array} \right\}$$

By Theorem 4.5, this polyhedron is integral, implying in turn that $x^\star$ is integral. This easily implies that $(x^\star, s^\star, t^\star)$ is also integral, thereby finishing the proof. □

Lemma 4.8. *Let A be a balanced matrix. Then the linear system $x, s, t \geq \mathbf{0}, Ax + Is - It = \mathbf{1}$ is totally dual integral.*

Proof. We prove this by induction on the number of rows of A. The base case is obvious. For the induction step, consider for integral weights b, c, d the primal program

$$(P) \qquad \begin{array}{ll} \max & b^\top x + c^\top s + d^\top t \\ \text{s.t.} & Ax + Is - It = \mathbf{1} \\ & x, s, t \geq \mathbf{0} \end{array}$$

and the dual

$$(D) \qquad \begin{array}{lll} \min & \mathbf{1}^\top y & \\ \text{s.t.} & A^\top y & \geq b \\ & y & \geq c \\ & -y & \geq d. \end{array}$$

We will construct an integral optimal solution to (D). To this end, take an optimal solution $\bar{y}$ to (D). If $\bar{y}$ is integral, we are done. Otherwise, we may assume that $\bar{y}_1$ is fractional. Write $\bar{y} = (\bar{y}_1, \bar{z})$. Let a be the first row of A, and let A' (resp. c', d') be the matrix (resp. vector) obtained from A (resp. c, d) after removing the first row. Consider the program

$$(D') \qquad \begin{array}{lll} \min & \mathbf{1}^\top z & \\ \text{s.t.} & A'^\top z & \geq b - \lceil \bar{y}_1 \rceil a \\ & z & \geq c' \\ & -z & \geq d'. \end{array}$$

Since $\bar{y} = (\bar{y}_1, \bar{z})$ is feasible for (D), we get that $\bar{z}$ is feasible for (D'). Our induction hypothesis implies that (D') has an integral optimal solution $z^\star$. In particular,

$$\mathbf{1}^\top \bar{z} \geq \mathbf{1}^\top z^\star.$$

As $z^\star$ is feasible for (D'), and c, d are integral, it follows that $(\lceil \bar{y}_1 \rceil, z^\star)$ is feasible for (D), so

$$\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star \geq \mathbf{1}^\top \bar{y} = \bar{y}_1 + \mathbf{1}^\top \bar{z}.$$

Combining the preceding two inequalities yields

$$\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star \geq \mathbf{1}^\top \bar{y} \geq \bar{y}_1 + \mathbf{1}^\top z^\star.$$

By Lemma 4.7, (P) has an integral optimal solution, so as b, c, d are integral, (P) has an integer optimal value. Thus, $\mathbf{1}^\top \bar{y}$ is an integer by LP Strong Duality. Hence, the inequalities above imply that $\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star = \mathbf{1}^\top \bar{y}$, so $(\lceil \bar{y}_1 \rceil, z^\star)$ is an integral optimal solution for (D), as required. This completes the induction step. $\square$

We are now ready to prove the following generalization of Theorem 4.6:

Theorem 4.9 (Conforti, Cornuéjols, Kapoor, Vušković 1996 [2]). *Let $G = (V, E)$ be a balanced hypergraph. Then the following statements are equivalent:*

- *G has no perfect matching,*
- *there are disjoint vertex sets R, B such that $|R| > |B|$ and for every edge e, $|e \cap B| \geq |e \cap R|$.*

Proof. ($\Leftarrow$) Suppose for a contradiction that G has a perfect matching $e_1, \ldots, e_k$. Then

$$|R| = \sum_{i=1}^{k} |e_i \cap R| \leq \sum_{i=1}^{k} |e_i \cap B| = |B| < |R|,$$

a contradiction. ($\Rightarrow$) Suppose G has no perfect matching. Let A be the vertex-edge incidence matrix of G. Notice that A is a balanced matrix. Consider the linear program

$$(P) \qquad \begin{array}{ll} \max & \mathbf{0}^\top x - \mathbf{1}^\top s - \mathbf{1}^\top t \\ \text{s.t.} & Ax + Is - It = \mathbf{1} \\ & x, s, t \geq \mathbf{0} \end{array}$$

Since G has no perfect matching, (P) has no integer feasible solution of value ≥ 0. It therefore follows from Lemma 4.7 that the optimal value of (P) is < 0. As a result, by Lemma 4.8, the dual program has an integral feasible solution of negative value, that is, there is an integral point $\bar{y}$ such that

$$\begin{aligned} \mathbf{1}^\top y &< 0 \\ A^\top y &\geq \mathbf{0} \\ y &\leq \mathbf{1} \\ y &\geq -\mathbf{1} \end{aligned}$$

Let $B := \{v \in V : \bar{y}_v = 1\}$ and $R := \{v \in V : \bar{y}_v = -1\}$. Clearly, $B \cap R = \emptyset$. The first inequality implies that $|R| > |B|$ while the second inequality implies that, for each edge e, $|e \cap B| \geq |e \cap R|$, as required. $\square$

This result has a nice Kőnig-type consequence. Given a hypergraph, the *degree* of a vertex is the number of edges containing that vertex. For an integer $d \geq 1$, a hypergraph is *d-regular* if every vertex has degree d.

Corollary 4.10. *The edges of a balanced hypergraph with maximum degree d can be partitioned into d matchings.*

Proof. Let $G = (V, E)$ be a balanced hypergraph with maximum degree $d \geq 1$. Let us first prove the result for d-regular hypergraphs:

Claim 1. *If G is d-regular, then its edges can be partitioned into d perfect matchings.*

Proof of Claim. We prove this by induction on $d \geq 1$. The base case $d = 1$ is obvious. Assume that $d \geq 2$. Let us use Theorem 4.9 to find a perfect matching in G. Take disjoint vertex subsets R, B of V such that for every edge e, $|e \cap B| \geq |e \cap R|$. Then

$$d \cdot |B| = \sum_{e \in E} |e \cap B| \geq \sum_{e \in E} |e \cap R| = d \cdot |R|,$$

implying in turn that $|B| \geq |R|$. It therefore follows from Theorem 4.9 that G has a perfect matching $M_d \subseteq E$. Notice that $G \setminus M_d$ is $(d-1)$-regular, so by the induction hypothesis, the edges of $G \setminus M_d$ can be partitioned into $d - 1$ perfect matchings $M_1, \ldots, M_{d-1}$. Together with M_d, we get a partition of the edges of G into d perfect matchings, thereby completing the induction step. $\diamondsuit$

Claim 2. *There is a d-regular balanced hypergraph $H = (V, E')$ such that $E \subseteq E'$.*

Proof of Claim. To obtain H, for every vertex v of G, add $d - \deg(v)$ edges of the form $\{v\}$. It is clear that H is a d-regular hypergraph. It is easy to see that H is a balanced hypergraph. $\diamondsuit$

By Claim 1, the edges of H can be partitioned into d perfect matchings. It is easy to see that this corresponds to a partition of the edges of G into d matchings, thereby finishing the proof. $\square$

In particular,

Theorem 4.11 (Kőnig 1931 [5])**.** *Let G be a bipartite graph of maximum degree d. Then the edges of G can be partitioned into d matchings, that is, G can be d-edge-colored.*

5 Perfect graphs

Let $G = (V, E)$ be a simple graph. Denote by $\chi(G)$ the minimum number of stable sets needed to cover V. Notice that $\chi(G)$ records the *chromatic number of* G, i.e. the minimum number of colors needed for a proper vertex-coloring. Denote by $\omega(G)$ the maximum cardinality of a clique. Since the vertices of a clique all get different colors in any proper vertex-coloring, it follows that

$$\chi(G) \geq \omega(G).$$

Denote by $\overline{G}$ the *complement* of G, that is, $\overline{G}$ has vertex set V where distinct vertices u, v are adjacent in $\overline{G}$ if they are non-adjacent in G. Notice that the cliques and stable sets of $\overline{G}$ are precisely the stable sets and cliques of $\overline{G}$, respectively.

Remark 5.1. *Let $G = (V, E)$ be a simple graph. Then*

$$\theta(G) := \chi(\overline{G})$$

is the minimum number of cliques of G needed to cover V, and

$$\alpha(G) := \omega(\overline{G})$$

is the maximum cardinality of a stable set. In particular, $\theta(G) \geq \alpha(G)$.

Recall the following result from Assignment 1:

Theorem 5.2 (Kőnig 1931 [5])**.** *In a bipartite graph, the minimum cardinality of a vertex cover is equal to the maximum cardinality of a matching.*

We will need this result moving forward, as well as a couple of notions. The *line graph* of a simple graph G is the graph on vertex set $E(G)$ where distinct $e, f \in E(G)$ are adjacent if e, f share a vertex of G. Given a partially ordered set $(V, \leq)$, its *comparability graph* is the graph on vertex set V where distinct $u, v \in V$ are adjacent if they are comparable.

The main theme of this section is, when does equality hold in $\chi \geq \omega$?

Theorem 5.3. *$\chi(G) = \omega(G)$ if G is any of the following graphs:*

(1) G or $\overline{G}$ is bipartite,

(2) G or $\overline{G}$ is the line graph of a bipartite graph,

(3) G or $\overline{G}$ is a comparability graph.

Proof. **(1)** Let G be a bipartite graph. Then $\chi(G) = 2 = \omega(G)$. We need to show that $\theta(G) = \alpha(G)$. Clearly,

$$\alpha(G) = |V| - k$$

where k is the minimum cardinality of a vertex cover. Since G is bipartite,

$$\theta(G) = |V| - m$$

where m is the maximum cardinality of a matching. By Theorem 5.2, $m = k$, implying in turn that $\theta(G) = \alpha(G)$, as required. **(2)** Let G be the line graph of a bipartite graph H. Observe that the stable sets and cliques of G are in correspondence with the matchings and stars of H, respectively. Thus $\chi(G)$ is equal to the minimum number of colors needed in an edge-coloring of H, while $\omega(G)$ is equal to the maximum degree of a vertex of H. It therefore follows from Theorem 4.11 that $\chi(G) = \omega(G)$. Moreover, $\theta(G)$ is equal to the minimum cardinality of a vertex cover of H, while $\alpha(G)$ is equal to the maximum cardinality of a matching of H. So by Theorem 5.2, $\theta(G) = \alpha(G)$. **(3)** Let $G = (V, E)$ be the comparability graph of a partially ordered set $(V, \leq)$. Then the cliques and stable sets of G are in correspondence with the chains and antichains of $(V, \leq)$. It therefore follows from Theorem 1.3 that $\theta(G) = \alpha(G)$, and it follows from Theorem 1.4 that $\chi(G) = \omega(G)$. □

Equality does not always hold in $\chi \geq \omega$. For instance, for the odd circuit C_5 on five vertices, $\chi(C_5) = 3 > 2 = \omega(C_5)$. Can we characterize when equality does hold? Is this even a well-posed question? Let H be an arbitrary graph, and let $k := \chi(H) - \omega(H) \geq 0$. Let $C \subseteq V(H)$ be a maximum clique of H. Let G be the graph obtained from H after adding k vertices and just enough edges so as to grow C into a clique of cardinality $\omega(H) + k$. Notice now that $\chi(G) = \chi(H) = \omega(H) + k = \omega(G)$. Starting from an arbitrary graph, we just constructed a graph for which equality holds in $\chi \geq \omega$. This construction tells us that asking when equality holds in

$$\chi \geq \omega$$

is an ill-posed question. To make sure this construction is ruled out, we will come up with a stronger notion.

Let $G = (V, E)$ be a simple graph. For $X \subseteq V$, the subgraph of G induced on vertices X is called an *induced subgraph* and is denoted $G[X]$. We say that G is *perfect* if, for every induced subgraph G' of G, $\chi(G') = \omega(G')$. (Notice that G' may be G.) In words, a simple graph is perfect if in each induced subgraph, the maximum cardinality of a clique is equal to the chromatic number. It follows from the preceding theorem that,

Corollary 5.4. *The following graphs are perfect:*

(1) bipartite graphs, and their complements,

(2) line graphs of bipartite graphs, and their complements,

(3) comparability graphs, and their complements.

The obvious question this corollary leads to is, does complementation preserve perfection? Claude Berge asked the same question in 1961 [1]. Although this may seem too good to be true, the answer is yes! We will prove this next time.

References

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