47853 Packing and Covering: Lecture 3

Ahmad Abdi

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4 Balanced matrices

Recall that an odd square matrix of the form

$$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{pmatrix}$$

with at least three columns is called an *odd circuit matrix* (all the other entries are set to 0). Recall that a $0-1$ matrix is *balanced* if it has no odd circuit submatrix, even after rearranging its rows and columns.

4.1 A bicoloring characterization of balanced matrices

A *bicoloring* of a $0-1$ matrix is a partition of the columns into two color classes, where every row with at least two 1s gets both colors. For instance, $R = \{1,4\}$ and $B = \{2,3\}$ yields a bicoloring of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

whose columns are labeled $1,2,3,4$ from left to right.

Theorem 4.1 (Berge 1972 [1])**.** *A $0-1$ matrix is balanced if, and only if, every submatrix has a bicoloring.*

Proof. Let A be a $0-1$ matrix. $(\Leftarrow)$ Since an odd circuit is not bipartite, an odd circuit matrix is not bicolorable. So, if every submatrix of A is bicolorable, A must be balanced. $(\Rightarrow)$ Suppose otherwise. We may assume that A is a balanced matrix that is not bicolorable, but every proper submatrix is bicolorable. In particular, every row of A has at least two 1s. Let V collect the column labels of A.

Claim. *For every $v \in V$, there exist rows of the form $\{v,u\},\{v,w\}$ for some distinct $u,w \in V - \{v\}$.*

Proof of Claim. For if not, bicolor the column submatrix of A corresponding to the columns $V - \{v\}$. Our contrary assumption allows us to extend this bicoloring to a bicoloring of A, a contradiction. $\Diamond$

Let G be the graph on vertices V whose edges correspond to the rows in A with exactly two 1s. Since A is balanced, and the edge-vertex incidence matrix of G is a submatrix of A, it follows that G is bipartite. By the claim above, every vertex of G has at least two distinct neighbors, so every connected component of G has at least four vertices. Pick a vertex v_0 of G that is not a cut-vertex. Now bicolor (R, B) the column submatrix of A corresponding to the columns $V - \{v_0\}$. Since G is bipartite, and v_0 is not a cut-vertex, the neighbors of v_0 get the same color, say R. Observe now that $(R, B \cup \{v_0\})$ is a bicoloring of A, a contradiction to our assumption. This finishes the proof of Theorem 4.1. $\square$

A *hypergraph* is a pair $G = (V, E)$ where V is a finite set of *vertices*, and each element of E is a nonempty subset of V, called an *edge*. A hypergraph is *balanced* if its incidence matrix is balanced.

Corollary 4.2 (Berge 1972 [1])**.** *Let $G = (V, E)$ be a balanced hypergraph, and let $k \geq 2$ be the minimum cardinality of an edge. Then there exists a partition of V into k color classes where every edge gets at least one vertex of each color.*

Proof. For $k = 2$, the result follows immediately from Theorem 4.1. We may therefore assume that $k \geq 3$. Let $(S_1, \dots, S_k)$ be an arbitrary partition of V. For each edge e, let

$$k_e := |\{i \in [k] : e \cap S_i \neq \emptyset\}| \in \{1, \dots, k\}.$$

If each k_e is k, then we have a k-coloring. Otherwise, assume that $k_g < k$ for some edge g. Since $|g| \geq k$, we may assume that

$$|g \cap S_{k-1}| \geq 2 \quad \text{and} \quad g \cap S_k = \emptyset.$$

Let A be the edge-vertex incidence matrix of G. Since A is balanced, by Theorem 4.1, we may bicolor the column submatrix of A corresponding to $S_{k-1} \cup S_k$ and get a bicoloring $S'_{k-1} \cup S'_k$. Consider now the partition $(S_1, \cdots, S_{k-2}, S'_{k-1}, S'_k)$. Notice that g intersects $k_g + 1$ many of these parts, and every other edge e intersects at least k_e many of these parts. By applying this argument recursively, we will achieve the desired k-coloring. $\square$

For an integer $k \geq 2$, a hypergraph is *k-partite* if its vertices can be partitioned into k parts such that every edge intersects each part at most once. As an immediate consequence of the preceding result, we have the following:

Corollary 4.3. *Take an integer $k \geq 2$ and a hypergraph where every edge has cardinality k. If G is balanced, then it is k-partite.*

4.2 Integral polyhedra associated with balanced matrices

Take a $0 - 1$ matrix A with column labels E, and consider the polytope

$$P(A) := \{\mathbf{1} \geq x \geq \mathbf{0} : Ax = \mathbf{1}\}.$$

Notice that for each $e \in E$,

$$P(A) \cap \{x : x_e = 0\} = P(A') \quad \text{and} \quad P(A) \cap \{x : x_e = 1\} = P(A'')$$

where A', A'' are appropriate submatrices of A. (Equality holds above after extending $P(A'), P(A'')$ to $\mathbb{R}^E$ by setting new coordinates to either 0 or 1.)

Proposition 4.4. *Let A be a balanced matrix. Then the polytope $P(A)$ is integral.*

Proof. Suppose otherwise. Let E be the column labels of A. We may assume that $P(A)$ is not integral, but for every proper submatrix A' of A, $P(A')$ is integral. In particular, for every $e \in E$, the two polytopes

$$P(A) \cap \{x : x_e = 0\} \quad \text{and} \quad P(A) \cap \{x : x_e = 1\}$$

are integral. Let $x^\star$ be a fractional extreme point of $P(A)$. Since the polytopes above are integral, it follows that $\mathbf{1} > x^\star > \mathbf{0}$. Our minimality assumption implies that A is a square nonsingular matrix.

Claim. *Every row of A has exactly two* 1*s.*

Proof of Claim. Since $\mathbf{1} > x^\star$, every row of A has at least two 1s. Let A' be the matrix obtained from A after removing the first row. Since $P(A')$ is integral and $x^\star \in P(A')$, it follows that $x^\star$ lies on an edge of $P(A')$. So for some vertices $\chi_S, \chi_T \in P(A')$ and $\lambda \in (0, 1)$,

$$x^\star = \lambda \chi_S + (1 - \lambda)\chi_T.$$

Since $\mathbf{1} > x^\star > \mathbf{0}$, it follows that $S \cap T = \emptyset$ and $S \cup T = E$. Since $A'\chi_S = \mathbf{1} = A'\chi_T$, every row of A other than the first row has exactly two 1s. A similar argument applied to the second row implies that even the first row has exactly two 1s. $\Diamond$

Since A is balanced, it is the incidence edge-vertex incidence matrix of a bipartite graph G. As A is a square matrix, G has an even circuit, which in turn contradicts the nonsingularity of A. This finishes the proof of Proposition 4.4. $\square$

Theorem 4.5 (Fulkerson, Hoffman, Oppenheim 1974 [2]). *Let $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ be a balanced matrix. Then the polyhedron*

$$P = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}, Bx \leq \mathbf{1}, Cx = \mathbf{1}\}$$

is integral. In particular, the set packing polytope and the set covering polyhedron corresponding to a balanced matrix are both integral.

Proof. Let $x^\star$ be an extreme point of P. Observe that $x^\star \leq \mathbf{1}$, and that $x^\star$ is also an extreme point of the polytope $\{\mathbf{1} \geq x \geq \mathbf{0} : Dx = \mathbf{1}\}$, where D is the row submatrix of $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ corresponding to the constraints of

$Ax \geq \mathbf{1}, Bx \leq \mathbf{1}, Cx = \mathbf{1}$ that are tight at $x^\star$. Since $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is balanced, so is D, so by Proposition 4.4, $x^\star$ is integral, as required. $\square$

In fact, the linear system above is totally dual integral. We will prove a similar result next time.

References

[1] Berge, C.: Balanced matrices. Math. Program. **2**(1), 19–31 (1972)

[2] Fulkerson, D.R., Hoffman, A., Oppenheim, R.: On balanced matrices. Math. Prog. Study **1**, 120–132 (1974)