47853 Packing and Covering: Lecture 2

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2 A review of integral polyhedra and totally dual integral linear systems

Take integers $m, n \geq 1$, a rational $m \times n$ matrix M, and a rational m-dimensional (column) vector b. The set

$$P := \{x \in \mathbb{R}^n : Mx \geq b\}$$

is called a *polyhedron*. If P is a bounded set, then it is called a *polytope*. If P does not contain a line, then it is *pointed*. For instances, polytopes are pointed, as well as polyhedra contained in the nonnegative orthant.

If every face of P contains an integral point, then P is an *integral* polyhedron.

Theorem 2.1 (Hoffman 1974 [5], Edmonds and Giles 1977 [3])**.** *Let P be a polyhedron. Then the following statements are equivalent:*

- *P is integral,*
- *for all $w \in \mathbb{Z}^n$, the program $\min\{w^\top x : x \in P\}$, if feasible and finite, has an integral optimal solution,*
- *for all $w \in \mathbb{Z}^n$, $\min\{w^\top x : x \in P\} \in \mathbb{Z} \cup \{\pm\infty\}$.*

For a variable cost vector $w \in \mathbb{Z}^n$, consider the primal linear program

$$(P) \qquad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & Mx \geq b \end{array}$$

and the dual linear program

$$(D) \qquad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & M^\top y = w \\ & y \geq \mathbf{0}. \end{array}$$

Here and throughout the rest of the document, $\mathbf{0}$ denotes the all-zeros vector of appropriate dimension. By LP Strong Duality, the optimal values of these two programs are equal, whenever the primal (P) is feasible and has a finite optimum. We say that the linear system $Mx \geq b$ is *totally dual integral (TDI)* if, for all $w \in \mathbb{Z}^n$ for which the primal (P) is feasible and has a finite optimum, the dual (D) has an integral optimal solution. The following is a consequence of Theorem 2.1:

Corollary 2.2. *If $Mx \geq b$ is a rational totally dual integral linear system and b is integral, then $\{x : Mx \geq b\}$ is an integral polyhedron.*

Proof. Assume that $Mx \geq b$ is a TDI linear system with n variables, and b is integral. Pick $w \in \mathbb{Z}^n$ such that (P) is feasible and has a finite optimum. Since $Mx \geq b$ is TDI, (D) has an integral optimal solution, and since b is integral, (D) has an integer optimal value, implying by LP Strong Duality that (P) has an integer optimal value. It now follows from Theorem 2.1 that $\{x : Mx \geq b\}$ is an integral polyhedron. $\square$

As an immediate consequence of Theorem 2.1 and Corollary 2.2,

Corollary 2.3. *Let $Mx \geq b$ be a rational linear system, where b is integral. Then the following statements are equivalent:*

- *$Mx \geq b$ is totally dual integral,*
- *for all $w \in \mathbb{Z}^n$ for which the primal (P) is feasible and has a finite optimum, both the primal (P) and the dual (D) have integral optimal solutions.*

We will always be working with nonempty, full-dimensional and pointed polyhedra P. For such polyhedra, integrality has a better definition. A *vertex*, or a *basic feasible solution* or an *extreme point*, of P is a point $x^\star \in P$ satisfying any of the following equivalent conditions:

- if for $x_1, x_2 \in P$ we have $x^\star = \frac{1}{2}x_1 + \frac{1}{2}x_2$, then $x_1 = x_2 = x^\star$,
- there is a row subsystem $M'x \geq b'$ of $Mx \geq b$ where $\operatorname{rank}(M') = n$ and $M'x^\star = b'$,
- there exists an integral cost vector $w \in \mathbb{Z}^n$ such that $x^\star$ is the unique optimal solution to the linear program
$$\min\left\{w^\top x : x \in P\right\}.$$

Theorem 2.4 (see [2], Theorem 3.33)**.** *Let P be a nonempty, full-dimensional and pointed polyhedron. Then P is integral if, and only if, every vertex is integral.*

3 Packing and covering models

There are mainly two polyhedra that we are interested in. Let A, B be $0-1$ matrices, where B has no column of all zeros. We will call
$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$
the *set covering polyhedron*, and
$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$
the *set packing polytope*. Here, $\mathbf{1}$ is the all-ones vectors of appropriate dimension. When are these polyhedra integral? When are the associated linear systems TDI? These questions will form the underlying theme of the entire course. The short answers are, the questions have been answered for the set packing case, and they are widely open for the set covering case. But first, why are we even interested?

3.1 The set covering polyhedron

Let A be a $0-1$ matrix. Consider the *set covering program*

$$(P) \qquad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & Ax \geq \mathbf{1} \\ & x \geq \mathbf{0} \end{array}$$

and its dual

$$(D) \qquad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \leq w \\ & y \geq \mathbf{0} \end{array}$$

for an integral cost vector w.[1] Notice that if w has a negative entry, then (P) does not have a finite optimum. We may therefore focus on nonnegative cost vectors w.

Packing st-paths. Let $G=(V,E)$ be a graph and take distinct vertices s,t. Let A be the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of st-paths. Let $w \in \mathbb{Z}_+^E$. Then the set covering program (P) can be rewritten as

$$\begin{array}{ll} \min & \sum (w_e x_e : e \in E) \\ \text{s.t.} & \sum (x_e : e \in P) \geq 1 \quad \forall\ st\text{-paths } P \\ & x_e \geq 0 \quad \forall e \in E. \end{array}$$

Note that every st-cut gives a feasible solution to (P). In particular, the minimum *weight* of an st-cut is an upper-bound on the optimal value of (P). Let G_w be the graph obtained from G after replacing each edge e by w_e parallel edges. Then the minimum weight of an st-cut in G is simply the minimum cardinality of an st-cut in G_w. Consider now the dual program (D), which may be rewritten as

$$\begin{array}{ll} \max & \sum (y_P : P \text{ is an } st\text{-path}) \\ \text{s.t.} & \sum (y_P : P \text{ is an } st\text{-path such that } e \in P) \leq w_e \quad \forall e \in E \\ & y_P \geq 0 \quad \forall\ st\text{-paths } P. \end{array}$$

Then a packing of st-paths in G_w gives a feasible solution to (D). We will think of a packing of st-paths in G_w as a *weighted* packing of st-paths in G (where each edge e appears in at most w_e many st-paths, and where an st-path may be packed more than once). Hence, the maximum value of a weighted packing of st-paths in G is a lower-bound on the optimal value of (D). It therefore follows from Theorem 1.1 that,

Corollary 3.1. *Let G be a graph and take distinct vertices s,t. Then the set covering system corresponding to the st-paths of G is totally dual integral. In particular, the set covering polyhedron*

$$\left\{x \in \mathbb{R}_+^E : \sum (x_e : e \in P) \geq 1 \quad \forall\ st\text{-paths } P\right\}$$

is integral.

[1] Fulkerson 1971 [4] called this dual LP the "packing program" for reasons that will become clear soon. However, in the current literature, the primal LP is referred to as the "set covering program". One possible explanation for this will be provided in the next chapter.

3.2 The set packing polytope

Let B be a $0-1$ matrix without a column of all zeros. Consider the *set packing program*

$$
(P) \qquad \begin{array}{ll} \max & w^\top x \\ \text{s.t.} & Bx \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array}
$$

and its dual

$$
(D) \qquad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & B^\top y \geq w \\ & y \geq \mathbf{0} \end{array}
$$

for an integral cost vector w.[2] Notice that if w has a negative entry, then the corresponding variable in an optimal solution will always be set to 0. We may therefore focus on nonnegative cost vectors w.

Covering with chains. Let $(E, \leq)$ be a partially ordered set. Let B be the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of chains. Then the set packing program (P) can be rewritten as

$$
\begin{array}{ll} \max & \sum (w_e x_e : e \in E) \\ \text{s.t.} & \sum (x_e : e \in C) \leq 1 \quad \forall \text{ chains } C \\ & x_e \geq 0 \quad \forall e \in E. \end{array}
$$

Observe that an antichain gives a feasible solution to (P). In particular, the maximum *weight* of an antichain is a lower-bound on the optimal value of (P). Let $(E_w, \leq)$ be the partially ordered set obtained from $(E, \leq)$ after replacing each element e by w_e pairwise incomparable copies. Then the maximum weight of an antichain of $(E, \leq)$ is simply the maximum cardinality of an antichain of $(E_w, \leq)$. Consider now the dual program (D), rewritten as

$$
\begin{array}{ll} \min & \sum (y_C : C \text{ is a chain}) \\ \text{s.t.} & \sum (y_C : C \text{ is a chain such that } e \in C) \geq w_e \quad \forall e \in E \\ & y_C \geq 0 \quad \forall \text{ chains } C. \end{array}
$$

Then a covering of E_w with chains gives a feasible solution to (D). We will think of a covering of E_w with chains as a *weighted* covering of E with chains (where each element e is covered at least w_e times, and chains can be used in a covering more than once). Thus, the minimum value of a weighted covering of E with chains is an upper-bound on the optimal value of (D). It therefore follows from Theorem 1.3 that,

Corollary 3.2. *Let $(E, \leq)$ be a partially ordered set. Then the set packing system corresponding to the chains of $(E, \leq)$ is totally dual integral. In particular, the set packing polytope*

$$
\left\{ x \in \mathbb{R}^E_+ : \sum (x_e : e \in C) \leq 1 \quad \forall \textit{ chains } C \right\}
$$

is integral.

[2] Fulkerson 1971 [4] called this dual LP the "covering program".

4 Balanced matrices

Let A, B be $0-1$ matrices, where B has no column of all zeros. Why is

$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$

called the set covering polyhedron and

$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$

the set packing polytope? There is a neat way to look at these polyhedra that explains the terminology and gives us a good intuition about what is coming. Take a graph $G = (V, E)$. Let A be the edge-vertex incidence matrix of G, that is, the columns are labeled by V and the rows are the incidence vectors of the edges. Then the $0-1$ points of

$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$

correspond to the vertex covers of G, hence the "set covering polyhedron". Let B be the vertex-edge incidence matrix of G, i.e. $B = A^\top$. Then the $0-1$ points of

$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$

correspond to the matchings of G, hence the "set packing polytope".

It follows from well-known theorems of Kőnig 1931 [6] that if G is bipartite, then the set covering and the set packing systems associated to the (edge-vertex or vertex-edge) incidence matrix are totally dual integral. Well, in general, we can think of any $0-1$ matrix as the (vertex-edge or edge-vertex) incidence matrix of a "hypergraph". How can we then generalize the notion of bipartiteness to hypergraphs? However way we do this, we want the definition to be invariant of taking matrix transpose.

An odd square matrix of the form

$$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{pmatrix}$$

with at least three columns is called an *odd circuit matrix*. A $0-1$ matrix is *balanced* if it has no odd circuit submatrix, even after rearranging its rows and columns. Observe that if a matrix is balanced, then so is its transpose. Notice that an odd circuit matrix is the incidence matrix of a graph odd circuit. As a result, the incidence matrix of a bipartite graph is always balanced. We may therefore think of balanced matrices as generalizations of bipartite graphs.

References

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