47853 Packing and Covering: Lecture 10

Ahmad Abdi

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8.2 T-joins and T-cuts

Let $G = (V, E)$ be a graph where loops and parallel edges are allowed; however, loops are thought of as vertex-less edges. For an edge subset $J \subseteq E$, denote by $\mathrm{odd}(J) \subseteq V$ the set of vertices incident with an odd number of edges of J. Clearly $\mathrm{odd}(J)$ has even cardinality. Notice that

$$\mathrm{odd}(J_1)\triangle\mathrm{odd}(J_2) = \mathrm{odd}(J_1\triangle J_2) \qquad J_1, J_2 \subseteq E,$$

where $\triangle$ is the symmetric difference operator. A subset $C \subseteq E$ is a *cycle* if $\mathrm{odd}(C) = \emptyset$. Observe that $\emptyset$ and loops are cycles. A *circuit* is a nonempty cycle that does not properly contain another nonempty cycle. We leave the following as an exercise:

Remark 8.6. *Let $G = (V, E)$ be a graph, and take a nonempty subset $C \subseteq E$. The C is a cycle if, and only if, C is a disjoint union of circuits.*

We will use this basic observation without reference. Take a subset $T \subseteq V$ of even cardinality. A *T-join* is an edge subset $J \subseteq E$ such that $\mathrm{odd}(J) = T$. For instance, $\emptyset$-joins are precisely cycles, and for distinct vertices $s, t \in V$, every st-path is an $\{s, t\}$-join.

Remark 8.7. *Take a graph $G = (V, E)$, a subset $T \subseteq V$ of even cardinality, and a T-join J. Then*

$$\{J' \subseteq E : J' \text{ is a } T\text{-join}\} = \{J\triangle C : C \text{ is a cycle}\}.$$

Proof. Suppose first that $J' \subseteq E$ is a T-join. Then $\mathrm{odd}(J'\triangle J) = \mathrm{odd}(J')\triangle\mathrm{odd}(J) = T\triangle T = \emptyset$, so $J'\triangle J$ is a cycle, and as $J' = J\triangle(J'\triangle J)$, we are done. Conversely, take a cycle C. Then $\mathrm{odd}(J\triangle C) = \mathrm{odd}(J)\triangle\mathrm{odd}(C) = T\triangle\emptyset = T$, so $J\triangle C$ is a T-join and we are done. □

Given a graph and a vertex subset T of even cardinality, what is the minimum cardinality of a T-join? When $T = \emptyset$, the answer is zero as $\emptyset$ is a T-join. We may therefore focus on nonempty T. The two remarks above provide the following partial answer to this question:

Remark 8.8 (Sebő 1987 [1])**.** *Take a graph $G = (V, E)$, a nonempty subset $T \subseteq V$ of even cardinality, and a T-join J. Define weights $w \in \{-1, 1\}^E$ as follows: for each $e \in J$ set $w_e := -1$, and for each $e \in E - J$ set $w_e := 1$. Then the following statements are equivalent:*

- *J is a minimum T-join,*
- *there is no cycle of total negative weight,*
- *there is no circuit of total negative weight.*

Take a graph $G = (V, E)$ and a *nonempty* subset $T \subseteq V$ of even cardinality. A *T-cut* is a cut of the form $\delta(U) \subseteq E$ where $|U \cap T|$ is odd. For instance, for distinct vertices s, t of G, an st-cut is an $\{s, t\}$-cut.

Proposition 8.9. *Take a graph $G = (V, E)$ and a nonempty subset $T \subseteq V$ of even cardinality. Let $\mathcal{C}$ be the clutter of minimal T-joins over ground set E. Then $b(\mathcal{C})$ is the clutter of minimal T-cuts.*

Proof. We need to show that (a) every T-cut is a cover of $\mathcal{C}$, and (b) every cover of $\mathcal{C}$ contains a T-cut. **(a)** Take a T-cut $\delta(U)$. We need to show that $\delta(U)$ intersects every T-join. Suppose otherwise. Take a T-join J such that $J \cap \delta(U) = \emptyset$. Then the odd-degree vertices of $J \cap E(G[U])$ are precisely $T \cap U$, a contradiction as $|T \cap U|$ is odd. **(b)** Conversely, let $B \subseteq E$ be a cover of $\mathcal{C}$. Then the graph $H := G \setminus B$ does not contain a T-join. To prove that B contains a T-cut of G, it suffices to argue why H has an empty T-cut. To this end, let A be the vertex-edge incidence matrix of H, and let $b \in \{0, 1\}^V$ be the incidence vector of $T \subseteq V$. (So the loops of H are the zero columns of A.) Since H has no T-join, it follows that the system

$$Ax \equiv b \pmod 2$$

has no $0 - 1$ solution. By the Farkas Lemma for binary spaces, there is a certificate $c \in \{0, 1\}^V$ such that

$$c^\top A \equiv \mathbf{0} \quad \text{and} \quad c^\top b \equiv 1 \pmod 2.$$

Pick $U \subseteq V$ such that $c = \chi_U$. The second equation implies that $|U \cap T|$ is odd, while the first equation implies that $\delta(U)$ is an empty cut of H, so $\delta(U)$ is an empty T-cut of H, as required. $\square$

Let's see what minors of the clutter of minimal T-joins correspond to in terms of the graph. Let $G = (V, E)$ be a graph and take a possibly empty subset $T \subseteq V$ of even cardinality. Let $\mathcal{C}$ be the clutter of minimal T-joins over ground set E. Take an edge $e \in E$. The *deletion* $(G, T) \setminus e$ is the pair $(G \setminus e, T)$. It is clear that the minimal T-joins of $(G, T) \setminus e$ are the members of $\mathcal{C} \setminus e$. The *contraction* $(G, T)/e$ is the pair $(G/e, T')$ where[1]

$$T' = \begin{cases} T - e & \text{if } |e \cap T| \text{ is even} \\ (T - e) \cup \{\text{shrunk vertex}\} & \text{if } |e \cap T| \text{ is odd.} \end{cases}$$

Observe that T' is a set of even cardinality. Notice that if J is a T-join of G, then $J - \{e\}$ is a T'-join of G/e. Conversely, if J' is a T'-join of G/e, then $J' \cup \{e\}$ contains a T-join of G. Hence, the minimal T'-joins of $(G, T)/e$ are the members of $\mathcal{C}/e$. For disjoint subsets $I, J \subseteq E$, the *minor* $(G, T) \setminus I/J$ is what is obtained after deleting I and contracting J. Notice that the minimal T'-joins of $(G \setminus I/J, T') := (G, T) \setminus I/J$ are the members of $\mathcal{C} \setminus I/J$.

[1] In this setting, to contract a loop is to delete it.

Let's get back to our question regarding minimum T-joins. Notice that the minimum cardinality of a T-join is equal to the covering number of the clutter of minimal T-cuts. So does the clutter of minimal T-cuts necessarily pack? Consider the complete graph K_4 on 4 vertices, let $T := V(K_4)$, and let $\mathcal{C}$ be its clutter of minimal T-cuts. Then $\mathcal{C}$ consists of the claws of K_4, and the blocker $b(\mathcal{C})$ – the minimal T-joins – consists of the claws as well as the perfect matchings. So $\tau(\mathcal{C}) = 2$, and as there are no disjoint claws, it follows that $\nu(\mathcal{C}) = 1$, so $\mathcal{C}$ does not pack. Despite this shortcoming, we can prove the following result. The proof we present is due to Sebő 1987 [1].

Theorem 8.10 (Seymour 1981 [2])**.** *Take a bipartite graph $G = (V, E)$, and a nonempty subset $T \subseteq V$ of even cardinality. Then the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts. That is, the clutter of minimal T-cuts of a bipartite graph packs.*

Proof. We proceed by induction on the number of vertices of G. The base case $|V| = 2$ holds trivially. For the induction step, assume that $|V| \geq 3$. Denote by τ the minimum cardinality of a T-join. We will construct τ disjoint T-cuts. If $\tau = 1$, then we are done. We may therefore assume that $\tau \geq 2$. Among all minimum T-joins, pick the one J whose longest path is the longest compared to the other ones. Define weights $w \in \{-1, 1\}^E$ as follows: for each $e \in J$ set $w_e := -1$, and for each $e \in E - J$ set $w_e := 1$. By Remark 8.8, G has no negative cycle, and as G is bipartite, every cycle has even weight.

Let Q be the longest path contained in J and let u, v be its ends. As Q is the longest path in J, and as G has no negative cycle, it follows that u, v each have degree 1 in J. In particular, $u, v \in \mathrm{odd}(J) = T$. Let $e^\star$ be the edge of Q incident with u. Then $J \cap \delta(u) = \{e^\star\}$.

Claim 1. *If C is a circuit such that $C \cap \delta(u) \neq \emptyset$ and $e^\star \notin C$, then $w(C) \geq 2$.*

Proof of Claim. Suppose otherwise. Since $w(C) \geq 0$ and $w(C)$ is even, it follows that $w(C) = 0$. So $J \triangle C$ is another minimum T-join, and as Q cannot be extended to a longer path in $J \triangle C$, Q and C must share a vertex other than u. Among all the vertices in $V(Q) - \{u\}$ that also belong to $V(C)$, pick the one w that is closest to u on Q. Let Q' be the uw-path in Q; as $e^\star \notin C$, it follows that $Q' \neq \emptyset$ and $Q' \cap C = \emptyset$. Let P_1, P_2 be the two uw-paths partitioning C. Since $w(P_1) + w(P_2) = w(C) = 0$ and $w(Q') < 0$, it follows that one of $P_1 \cup Q', P_2 \cup Q'$ is a negative circuit, a contradiction. ◊

Claim 2. *u cannot be adjacent to all the other vertices in T.*

Proof of Claim. Suppose otherwise. In particular, u and v are adjacent, and as G has no negative cycle, Q has length 1. Since Q is the longest path in J, it follows that J is a matching, and as $\tau \geq 2$, the matching has an edge other than the edge of Q. Since u is adjacent to the other matched vertices, G has a triangle, a contradiction as G is bipartite. ◊

Let $(G', T') := (G, T)/\delta(u)$. Notice that G' is still a bipartite graph, and by Claim 2, $T' \neq \emptyset$. Let $J' := J - \delta(u)$. Then J' is a T'-join of G' of length $\tau - 1$. In fact,

Claim 3. *J' is a minimum T'-join of G'.*

Proof of Claim. Define weights $w' \in \{-1,1\}^{E(G')}$ on the edges of G' as follows: for each $e \in J'$ set $w'(e) := -1$, and for each $e \in E(G') - J'$ set $w'(e) := 1$. Notice that w' is simply the restriction of w to $E - \delta(u) = E(G')$. To prove that J' is a minimum T'-join of G', it suffices by Remark 8.8 to show that G' does not have a negative circuit. To this end, let C' be a circuit of G', and let C be a circuit of G such that $C' \subseteq C \subseteq C' \cup \delta(u)$. If $C = C'$ or $e^\star \in C$, then $w'(C') = w(C) \geq 0$. Otherwise, $C \cap \delta(u) \neq \emptyset$ and $e^\star \notin C$. It therefore follows from Claim 1 that

$$w'(C') = w(C) - 2 \geq 0,$$

as required. $\diamondsuit$

Thus, by the induction hypothesis, G' has $\tau - 1$ disjoint T-cuts; these are also disjoint T-cuts of G, and together with $\delta(u)$, they give τ disjoint T-cuts in G, thereby completing the induction step. This finishes the proof. $\square$

References

[1] Sebő, A.: A quick proof of Seymour's theorem on T-joins. Discrete Math. **64**, 101–103 (1987)

[2] Seymour, P.D.: On odd cuts and plane multicommodity flows. Proc. London Math. Soc. **42**(1), 178–192 (1981)