47853 Packing and Covering: Lecture 1

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1 What is packing and covering?

1.1 Menger's theorem and its dual

Let $G = (V, E)$ be a graph, and take distinct vertices $s, t \in V$.[1] An *st-path* is a minimal edge subset connecting s and t. An *st-cut* is an edge subset of the form

$$\delta(U) := \{e \in E : |e \cap U| = 1\}$$

where $U \subseteq V$ satisfies $U \cap \{s, t\} = \{s\}$. We will refer to U and $V - U$ as the *shores* of G. Notice that every st-path intersects every st-cut.

What is the maximum number of (pairwise) disjoint st-paths? In other words, how many st-paths can we *pack*?

Theorem 1.1 (Menger 1927 [10])**.** *Let $G = (V, E)$ be a graph, and take distinct vertices $s, t \in V$. Then the maximum number of disjoint st-paths is equal to the minimum cardinality of an st-cut.*

Proof. Every st-path intersects an st-cut, so the maximum number of disjoint st-paths is at most the minimum cardinality of an st-cut. We prove the other inequality by induction on $|V| + |E| \geq 3$. The result is obvious for $|V| + |E| = 3$. For the induction step, assume that $|V| + |E| \geq 4$. Let τ be the minimum cardinality of an st-cut. We may assume that $\tau \geq 1$. We will find τ disjoint st-paths.

Claim 1. *If an edge e does not appear in a minimum st-cut, then G has τ disjoint st-paths.*

Proof of Claim. Notice that the cardinality of a minimum st-cut in $G \setminus e$ is still τ. As a result, the induction hypothesis implies the existence of τ disjoint st-paths in $G \setminus e$, and therefore in G. $\Diamond$

We may therefore assume that every edge appears in a minimum st-cut. An st-cut $\delta(U)$ is *trivial* if either $|U| = 1$ or $|V - U| = 1$.

Claim 2. *If there is a minimum st-cut that is not trivial, then G has τ disjoint st-paths.*

[1] We allow parallel edges but disallow loops, until further notice.

Proof of Claim. Let $\delta(U), s \in U \subseteq V - \{t\}$ be a minimum st-cut that is non-trivial. Let G_1 be the graph obtained from G by shrinking U to a single vertex s', and let G_2 be the graph obtained from G after shrinking $V - U$ to a single vertex t'. Since $\delta(U)$ is non-trivial, it follows that $|V(G_i)| + |E(G_i)| < |V| + |E|$, for each $i \in [2]$. We may therefore apply the induction hypothesis to G_1 and G_2. Notice that τ is still the minimum cardinality of an $s't$-cut in G_1 and of an st'-cut in G_2. Thus, by the induction hypothesis, G_1 has τ disjoint $s't$-paths and G_2 has disjoint st'-paths. Gluing these paths along the edges of $\delta(U)$ gives us τ disjoint st-paths in G. $\Diamond$

We may therefore assume that every minimum st-cut is trivial. Since every edge appears in a minimum st-cut, it follows that every edge has either s or t as an end. In this case, G has a special form and it is clear that $\tau = \nu$ for this graph, thereby completing the induction step. $\square$

On the other hand, how many st-cuts can we pack?

Theorem 1.2. *Let $G = (V, E)$ be a connected graph G, and take distinct vertices $s, t \in V$. Then the maximum number of disjoint st-cuts is equal to the minimum cardinality of an st-path.*

Proof. Clearly, the maximum number of disjoint st-cuts is at most the minimum cardinality of an st-path. To prove the other inequality, let $\tau \geq 1$ be the minimum cardinality of an st-path. We will find τ disjoint st-cuts. Notice that τ is equal to the distance between s and t. For each $i \in \{0, 1, \ldots, \tau - 1\}$, let U_i be the set of vertices at distance at most i from s. Notice that $\{s\} = U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_{\tau-1} \subseteq V - \{t\}$. Our definition implies that $\delta(U_0), \delta(U_1), \ldots, \delta(U_{\tau-1})$ are disjoint st-cuts, as required. $\square$

These results are two of many packing theorems. Just to mention a few, we will see some of these packing results:

- Lucchesi and Younger 1978 [9]: given a directed graph G, the maximum number of disjoint dicuts is equal to the minimum cardinality of a dijoin.
- Conjecture (Woodall 1978 [13]): given a directed graph G, the maximum number of disjoint dijoins is equal to the minimum cardinality of a dicut.
- Edmonds and Johnson 1973 [4]: given a graph G and even subset T of vertices, the maximum value of a fractional packing of T-joins is equal to the minimum cardinality of a T-cut.
- Guenin 2001 [7]: in a signed graph without an odd-K_5 minor, the maximum value of a fractional packing of odd circuits is equal to the minimum cardinality of a signature.

1.2 Dilworth's theorem and its dual

Take a partially ordered set $(E, \leq)$, that is, the following statements hold for all $a, b, c \in E$:

- $a \leq a$,

- if $a \leq b$ and $b \leq a$, then $a = b$,
- if $a \leq b$ and $b \leq c$, then $a \leq c$.

We say that a, b are *comparable* if $a \geq b$ or $b \geq a$; otherwise they are *incomparable*. A *chain* is a set of pairwise comparable elements. An *antichain* is a set of pairwise incomparable elements. Notice that every antichain intersects every chain at most once.

What is the minimum number of (not necessarily disjoint) chains whose union is E? That is, what is the least number of chains needed to *cover* the ground set?

Theorem 1.3 (Dilworth 1950 [2]). *Let $(E, \leq)$ be a partially ordered set. Then the minimum number of chains needed to cover E is equal to the maximum cardinality of an antichain.*

Proof. Since every chain intersects every antichain at most once, the minimum number of chains needed to cover E is greater than or equal to the maximum cardinality of an antichain. We will prove the other inequality by induction on $|E|$. The base case $|E| = 1$ is obvious. For the induction step, assume that $|E| \geq 2$. Let α be the maximum cardinality of an antichain. We will find α chains covering E. If $\alpha = |E|$, then we are clearly done. Otherwise, $\alpha < |E|$, implying in turn that there is a chain $\{a, b\}$ where a is a minimal element and b is a maximal element. Let $E' := E - \{a, b\}$.

Claim. *If the maximum cardinality of an antichain of $(E', \leq)$ is $\alpha - 1$, then there are α chains covering E.*

Proof of Claim. By the induction hypothesis, there are $\alpha - 1$ chains of E' covering $E - \{a, b\}$. Together with $\{a, b\}$, we get a covering of E using α chains. ♢

We may therefore assume that E' has an antichain A such that $|A| = \alpha$. Let

$$E^+ := A \cup \{x \in E - A : x \geq z \text{ for some } z \in A\}$$
$$E^- := A \cup \{y \in E - A : y \leq z \text{ for some } z \in A\}.$$

Since A is an antichain, $E^+ \cap E^- = A$, and since it is a maximum antichain, $E^+ \cup E^- = E$. As a is minimal and $a \notin A$, it follows that $a \notin E^+$. As b is maximal and $b \notin A$, we get that $b \notin E^-$. In particular, $|E^+|, |E^-| < |E|$. Thus, by the induction hypothesis, E^+ has α chains covering it, and E^- has α chains covering it. Gluing these chains together, we get α chains covering $E^+ \cup E^- = E$, thereby completing the induction step. □

On the other hand, what is the least number of antichains needed to cover the ground set?

Theorem 1.4. *Let $(E, \leq)$ be a partially ordered set. Then the minimum number of antichains needed to cover E is equal to the maximum cardinality of a chain.*

Proof. Clearly, the minimum number of antichains needed to cover E is greater than or equal to the maximum cardinality of a chain. To prove the other inequality, let α denote the maximum cardinality of a chain. We will find α antichains whose union is E. Let A_1 denote the set of all minimal elements of E. For each $i \geq 2$, let A_i denote the set of all minimal elements of $E - (A_1 \cup \cdots \cup A_{i-1})$. Observe that

- $E = \bigcup_{i \geq 1} A_i$,
- each A_i is an antichain,
- if $i \geq 2$ and $a \in A_i$, then there is a $b \in A_{i-1}$ such that $a \geq b$, and so
- if $A_i \neq \emptyset$, then there is a chain of cardinality i.

As a result, since α is the maximum cardinality of a chain, it follows that $\emptyset = A_{\alpha+1} = A_{\alpha+2} = \cdots$. Thus, E is the union of the α antichains $A_1, \ldots, A_\alpha$, as required. □

These results are two of many covering results. To name a few:

- Kőnig 1931 [8]: In a bipartite graph, the minimum number of colors needed for a proper edge-coloring is equal to the maximum degree of a vertex.
- Gallai 1962 [6], Surányi 1968 [12]: In a chordal graph, the minimum number of cliques needed to cover the vertices is equal to the maximum cardinality of a stable set.
- Sachs 1970 [11]: In a chordal graph, the minimum number of colors needed for a proper vertex-coloring is equal to the maximum cardinality of a clique.
- Chudnovsky, Robertson, Seymour and Thomas 2006 [1]: In a graph without an odd hole or an odd hole complement, the minimum number of cliques needed to cover the vertices is equal to the maximum cardinality of a stable set.

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