47853 Assignment 4

Due Tuesday, February 26, in class or by email

The assignment will be out of 35 points. If you get more than 35 points, the extra points will be added to your previous assignments, if needed, with half the weight.

Q1. (**8 points**) Let $G = (V, E)$ be a graph.

(a) Take a subset $\Sigma \subseteq E$. The pair (G, Σ) is called a *signed graph*. An *odd circuit* is a circuit C such that $|C \cap \Sigma|$ is odd. A *signature* is a set of the form $\Sigma \triangle \delta(U), U \subseteq V$.[1] Prove that the clutter of minimal signatures and the clutter of odd circuits are blockers.

(b) Let $\mathcal{C}$ be the clutter over ground set E whose members are the complements of spanning trees. Prove that $b(\mathcal{C})$ is the clutter of circuits.

Q2. (**5 points**) Let $\mathcal{C}$ be a clutter over ground set E and let $w \in \mathbb{R}^E$. Prove that

$$\max_{C \in \mathcal{C}} \min_{x \in C} w_x = \min_{B \in b(\mathcal{C})} \max_{x \in B} w_x.$$

Q3. (**7 points**) Let $G = (V, E)$ be a simple graph, and let $\mathcal{C}$ be the clutter of the minimal vertex covers of G. Prove that the following statements are equivalent:

(i) $\mathcal{C}$ is ideal,

(ii) G is bipartite,

(iii) $\mathcal{C}$ is Mengerian.

Q4. (**5 points**) Let $\mathcal{C}$ be a clutter whose incidence matrix is balanced. Prove that $b(\mathcal{C})$ is Mengerian.

Q5. (**5 points**) Let $\mathcal{C}$ be a clutter over ground set E. Prove that $\{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ is integral if, and only if, $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ is integral.

Q6. **(5 points)** Let D be a directed planar graph. A *feedback arc set* is an arc subset whose deletion makes the directed graph acyclic. Prove that the minimum cardinality of a feedback arc set is equal to the maximum number of arc-disjoint directed circuits.

[1] $A \triangle B = (A \cup B) - (A \cap B)$

Q7. **(5 points)** Let $G = (V, E)$ be a loopless graph where $|V|$ is even. Prove that the convex hull of

$$\{\chi_M : M \subseteq E \text{ is a perfect matching}\}$$

is equal to

$$\left\{\mathbf{1} \geq x \geq \mathbf{0} : \sum_{e \in E} x_e = \frac{|V|}{2}, \sum (x_e : e \in \delta(U)) \geq 1 \ \ \forall\, U \subseteq V, |U| \text{ is odd}\right\}.$$

Q8. **(5 points)** Take an integer $r \geq 3$, and take an r-regular graph $G = (V, E)$ where $|V|$ is even, and for each odd cardinality $U \subseteq V$, $|\delta(U)| \geq r$. Prove that every edge of G belongs to a perfect matching.